

Computational Geometry Algorithms Library

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<http://www.cgal.org/siggraph2009>

The updated handout has

- extensive comments in the “Notes” part of Powerpoint.
- hyperlinks to the CGAL User and Reference Manual (~3500 pages).
- hyperlinks to precompiled demos illustrating the algorithms. They will be made available online.

Course Outline

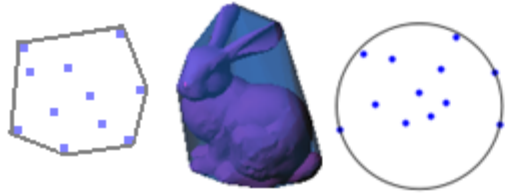
- General Introduction
- CGAL for 2D Vector Graphics
- CGAL for Point Sets
- CGAL for Modeling and Processing of Polyhedral Surfaces
- CGAL for Mesh Generation
- Questions and Answers

Mission Statement

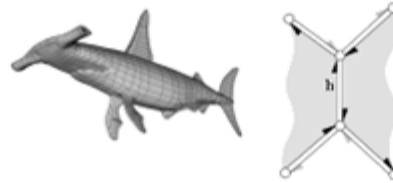
“Make the large body of geometric algorithms developed in the field of computational geometry available for industrial applications”

CGAL Project Proposal, 1996

Algorithms and Datastructures



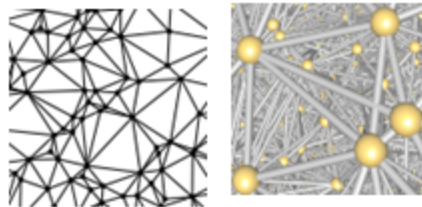
Bounding Volumes



Polyhedral Surface



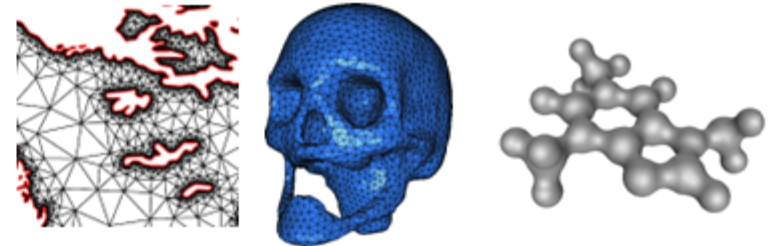
Boolean Operations



Triangulations



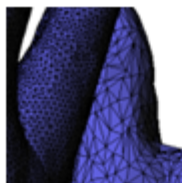
Voronoi Diagrams



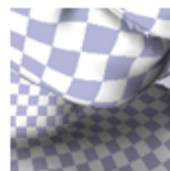
Mesh Generation



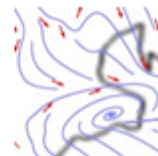
Subdivision



Simplification



Parameterization



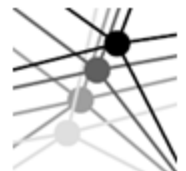
Streamlines



Ridge
Detection



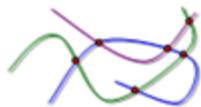
Neighbor
Search



Kinetic
Datastructures



Lower Envelope



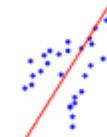
Arrangement



Intersection
Detection



Minkowski
Sum



PCA



Polytope
distance



QP Solver

CGAL in Numbers

500,000	lines of C++ code
10,000	downloads/year (+ Linux distributions)
3,500	manual pages
3,000	subscribers to cgal-announce
1,000	subscribers to cgal-discuss
120	packages
90	commercial users
20	active developers
12	months release cycle
2	licenses: Open Source and commercial

The diagram illustrates a central hub-and-spoke model for a research or industrial project. The central hub is a purple square with a white 'G' and a circuit symbol, representing a core technology or organization. This hub is connected to eight main functional areas: Image Processing, CAD/CAM, Medical, Scientific visualization, Geophysics (Oil&Gas), Telecom, GIS, and Digital maps. Surrounding this central hub are numerous logos of partner organizations, including VDRRC, NOESIS, Wihofszky Software, Exa Corporation, MPC, ZWCAD, ARADEX, ECL, DASSAULT SYSTEMES, LEUVEN, TOMTEC, GE Healthcare, ST. JUDE MEDICAL, The MathWorks, INDUSTRIAL RESEARCH LIMITED, TGS, Agip, Midland Valley, BSAP, TOTAL, brgm, france telecom, BT, TruePosition, WNI weathernews, ARCHI VIDEO, Polytec, SAFE SOFTWARE, Leica Geosystems, QinetiQ, virtualwind, K, rm DATA, ESRI, ENVITIA, BAE SYSTEMS, Agilent Technologies, cadence, TOSHIBA, ECT, Orbotech, pulsic, and Leica Geosystems.

Why They Use CGAL

“ I recommended to the senior management that we start a policy of buying-in as much functionality as possible to reduce the quantity of code that our development team would have to maintain.

This means that we can concentrate on the application layer and concentrate on our own problem domain.”

Senior Development Engineer
& Structural Geologist

Midland Valley Exploration

Why They Use CGAL

“ My research group JYAMITI at the Ohio State University uses CGAL because it provides an efficient and robust code for Delaunay triangulations and other primitive geometric predicates. Delaunay triangulation is the building block for many of the shape related computations that we do. [...]

Without the robust and efficient codes of CGAL, these codes could not have been developed. ”

Tamal Dey
Professor, Ohio State University

CGAL Open Source Project

Project = « Planned Undertaking »

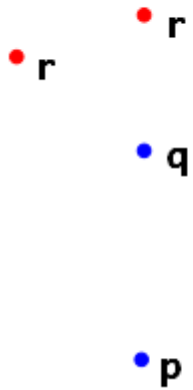
- Institutional members make a long term commitment: Inria, MPI, Tel-Aviv U, Utrecht U, Groningen U, ETHZ, GeometryFactory, FU Berlin, Forth, U Athens
- Editorial Board
 - Steers and animates the project
 - Reviews submissions
- Development Infrastructure
 - Gforge: svn, tracker, nightly testsuite,...
 - 120p developer manual and mailing list
 - Two 1-week developer meetings per year

Contributions

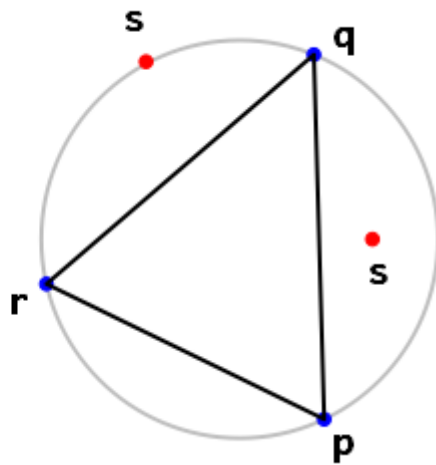
- Submission of specifications of new contributions
- Review and decision by the Editorial Board
- Value for contributor
 - Integration in the CGAL community
 - Gain visibility in a mature project
 - Publication value for accepted contributions

Exact Geometric Computing

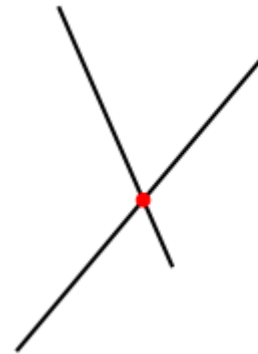
Predicates and Constructions



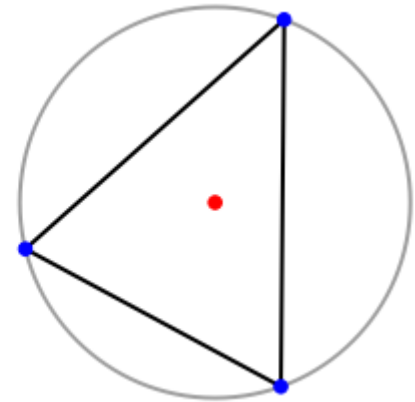
orientation



in_circle



intersection



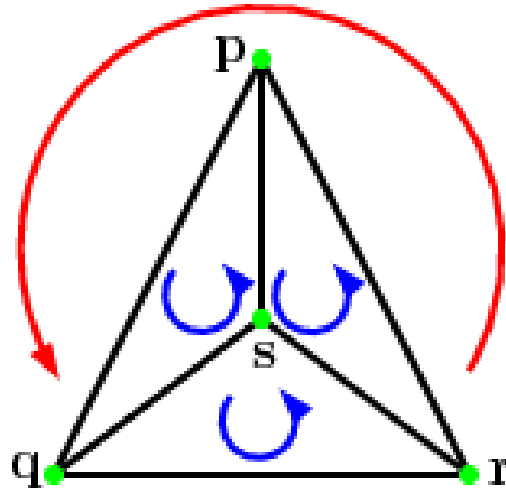
circumcenter

Robustness Issues

- Naive use of floating-point arithmetic causes geometric algorithms to:
 - Produce [slightly] wrong output
 - Crash after invariant violation
 - Infinite loop
- There is a gap between
 - Geometry in theory
 - Geometry with floating-point arithmetic

Geometry in Theory

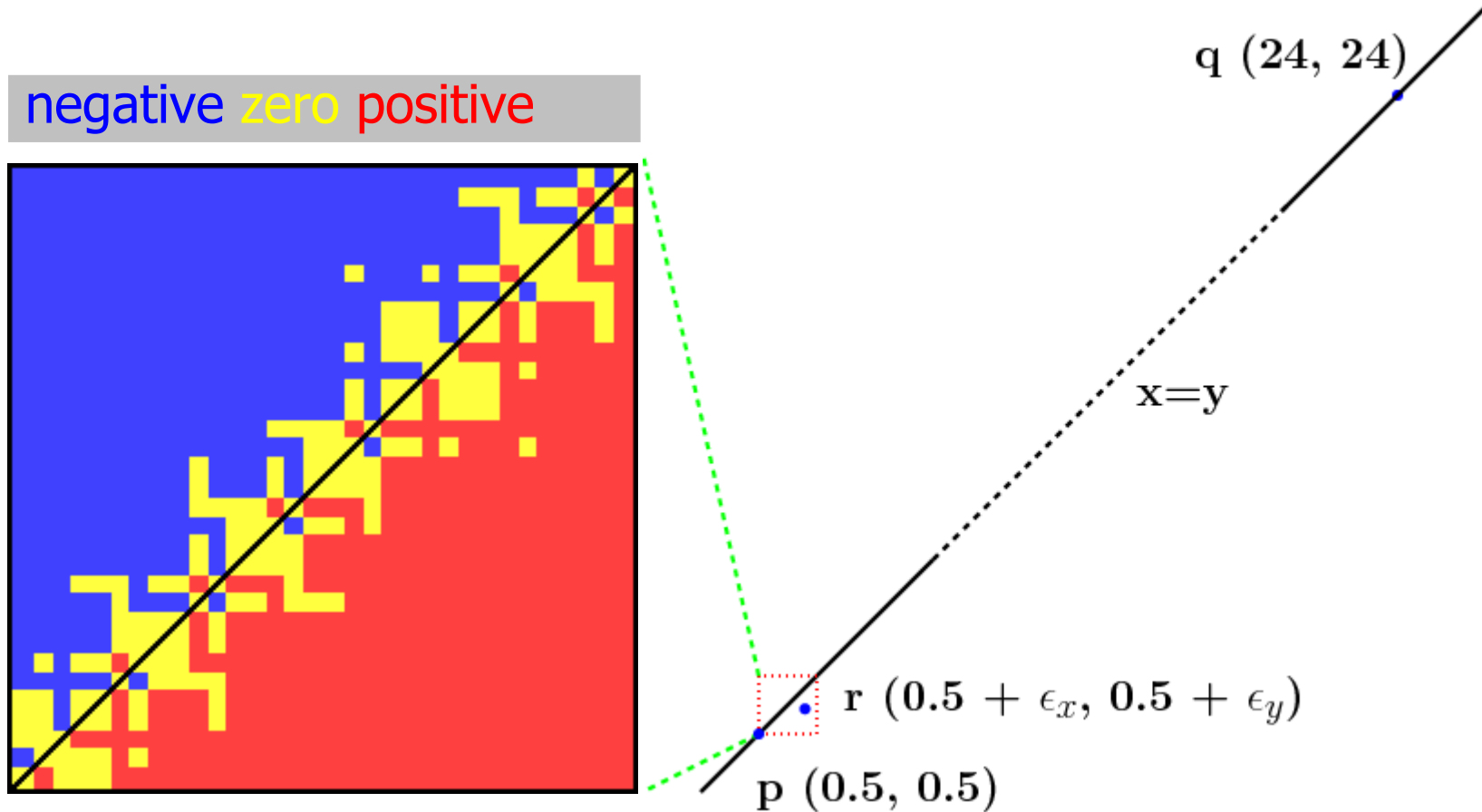
$$\text{ccw}(s,q,r) \ \& \ \text{ccw}(p,s,r) \ \& \ \text{ccw}(p,q,s) \Rightarrow \text{ccw}(p,q,r)$$



Correctness proofs of algorithms rely on such theorems

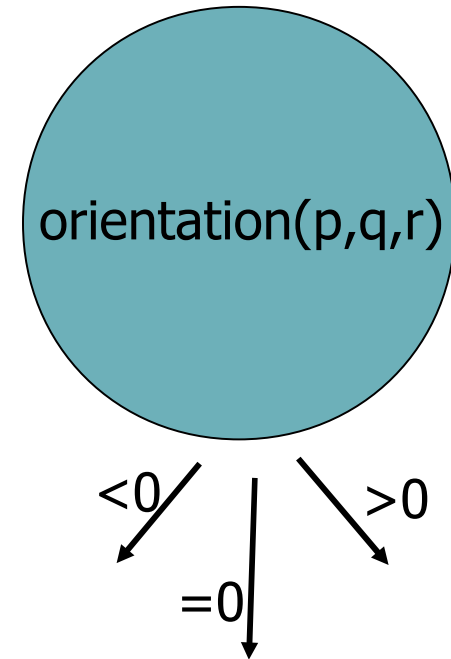
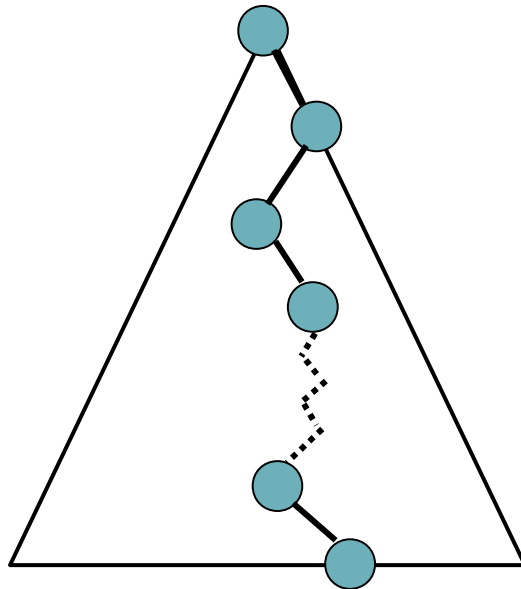
Demo: The Trouble with Double

$$\text{orientation}(p, q, r) = \text{sign}((p_x - r_x)(q_y - r_y) - (p_y - r_y)(q_x - r_x))$$



Exact Geometric Computing [Yap]

Make sure that the control flow in the implementation corresponds to the control flow with exact real arithmetic



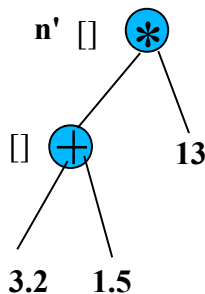
Filtered Predicates

- Generic functor adaptor `Filtered_predicate<>`
 - Try the predicate instantiated with intervals
 - In case of uncertainty, evaluate the predicate with multiple precision arithmetic
- Refinements:
 - Static error analysis
 - Progressively increase precision

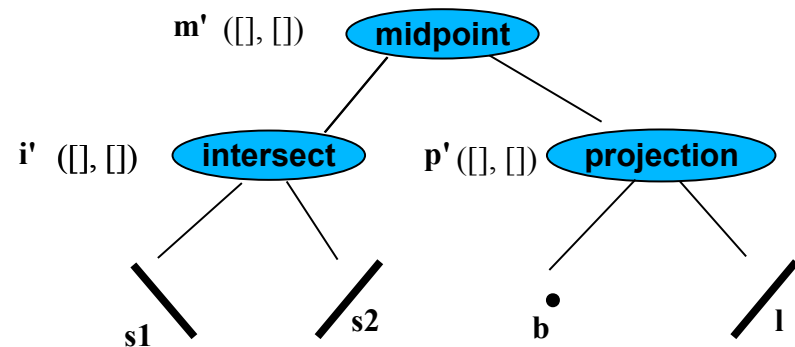
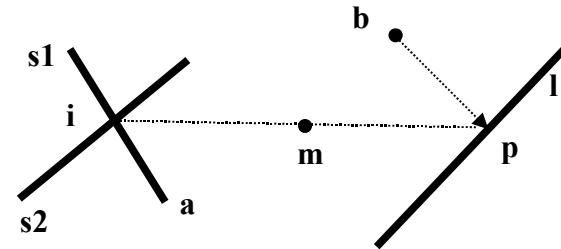
Filtered Constructions

Lazy number = interval and arithmetic expression tree

$$(3.2 + 1.5) * 13$$



Lazy object = approximated object and geometric operation tree



Test that may trigger an exact re-evaluation:

if ($n' < m'$)

if ($\text{collinear}(a', m', b')$)

The User Perspective

- Convenience Kernels
 - `Exact_predicates_inexact_constructions_kernel`
 - `Exact_predicates_exact_constructions_kernel`
 - `Exact_predicates_exact_constructions_kernel_with_sqrt`
- Number Types
 - `double`, `float`
 - `CGAL::Gmpq (rational)`, `Core (algebraic)`
 - `CGAL::Lazy_exact_nt<ExactNT>`
- Kernels
 - `CGAL::Cartesian<NT>`
 - `CGAL::Filtered_kernel<Kernel>`
 - `CGAL::Lazy_kernel<NT>`

Merits and Limitations

- Ultimate robustness inside the black box
- The time penalty is reasonable, e.g. 10% for 3D Delaunay triangulation of 1M random points
- Limitations of Exact Geometric Computing
 - Topology preserving rounding is non-trivial
 - Construction depth must be reasonable
 - Cannot handle trigonometric functions

Generic Programming

STL Genericity

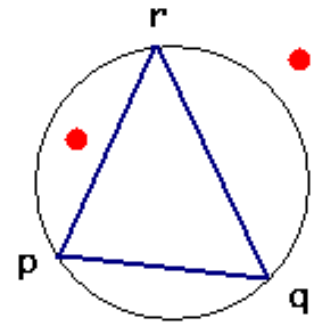
```
template <class Key, class Less>
class set {
    Less less;

    insert(Key k)
    {
        if (less(k, treenode.key))
            insertLeft(k);
        else
            insertRight(k);
    }
};
```

CGAL Genericity

```
template < class Geometry >
class Delaunay_triangulation_2 {
    Geometry::Orientation orientation;
    Geometry::In_circle in_circle;

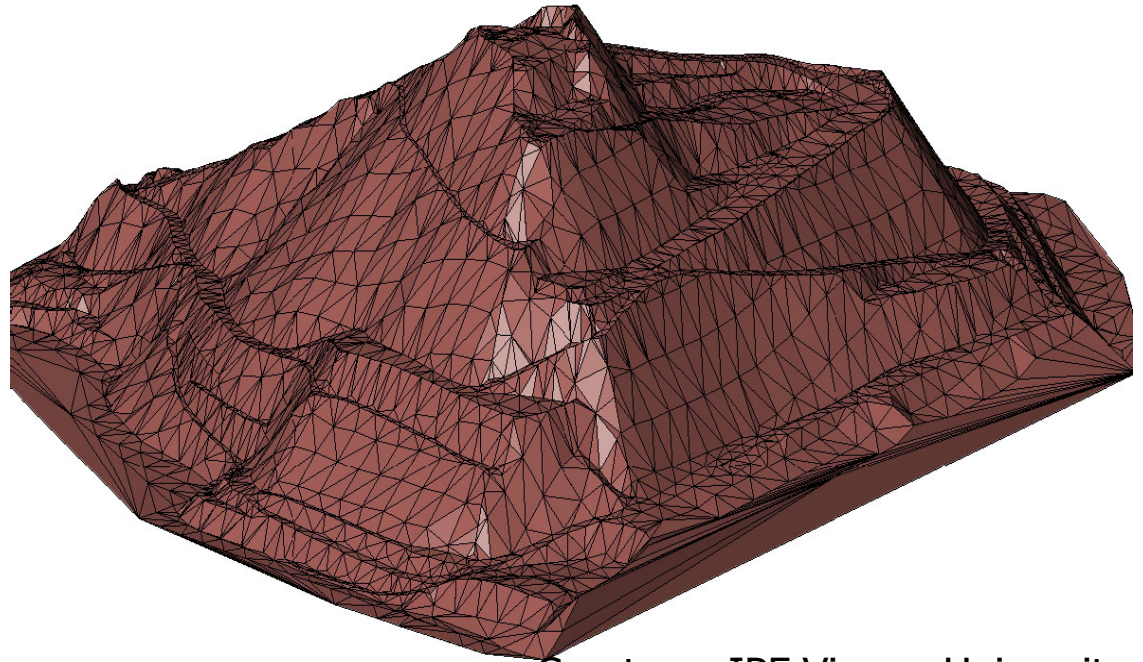
    void insert(Geometry::Point t) {
        ...
        if(in_circle(p,q,r,t)) {...}
        ...
        if(orientation(p,q,r){...}
    }
};
```



CGAL Genericity Demo

Without explicit conversion to points in the plane

- Triangulate the terrain in an xy-plane
- Triangulate the faces of a Polyhedron

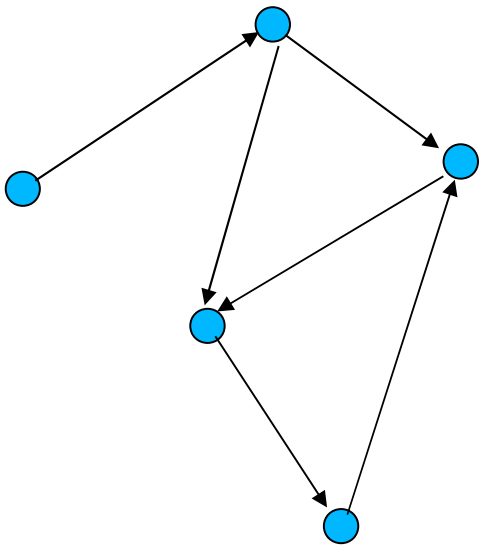


Courtesy: IPF, Vienna University
of Technology & Inpho GmbH

Boost Graph Library (BGL)

- Rich collection of graph algorithms:
shortest paths, minimum spanning tree, flow, etc.
- Design that
 - decouples data structure from algorithm
 - links them through a thin glue layer
- BGL and CGAL
 - Provide glue layer for CGAL
 - Extension to embedded graphs
inducing the notion of faces

BGL Glue Layer: Traits Class



```
template <typename Graph >
struct boost::graph_traits {
    typedef ... vertex_descriptor;
    typedef ... edge_descriptor;
    typedef ... vertex_iterator;
    typedef ... out_edge_iterator;
};
```

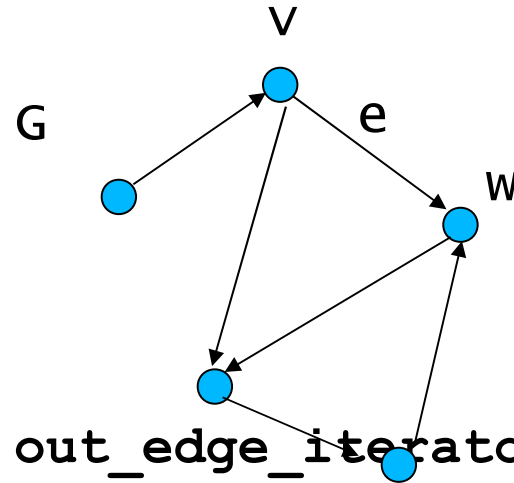

BGL Glue Layer: Free Functions

```
vertex_descriptor v, w;  
edge_descriptor e;
```

```
v = source(e,G) ;  
w = target(e,G) ;
```

```
std::pair<out_edge_iterator, out_edge_iterator> ipair;
```

```
ipair = out_edges(v,G) ;
```



BGL Glue Layer for CGAL

CGAL provides partial specializations:

```
template <typename T>  
graph_traits<Polyhedron<T>>;
```

```
template <typename T>  
Polyhedron<T>::Vertex  
source (Polyhedron<T>::Edge) ;
```

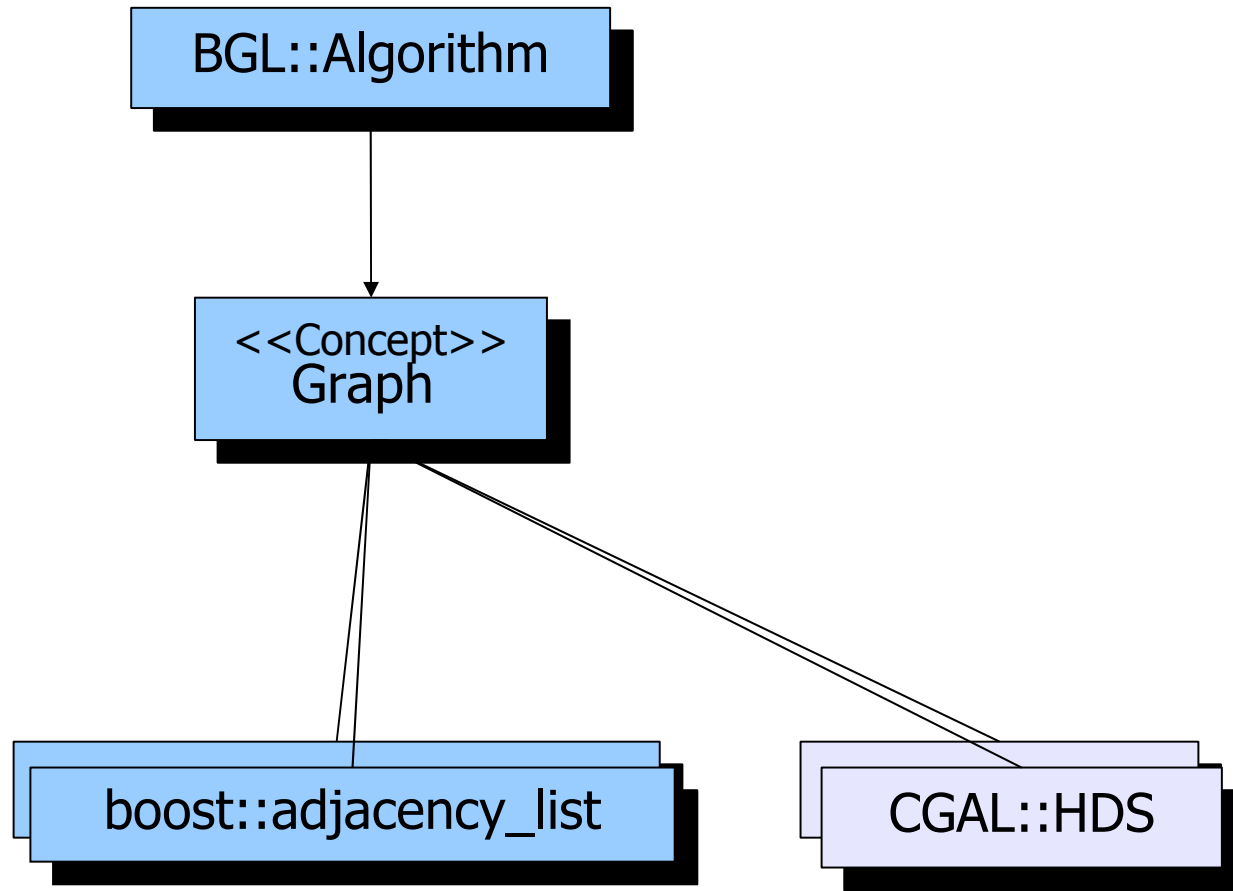
Users can run:

```
boost::kruskal_mst(P) ;
```

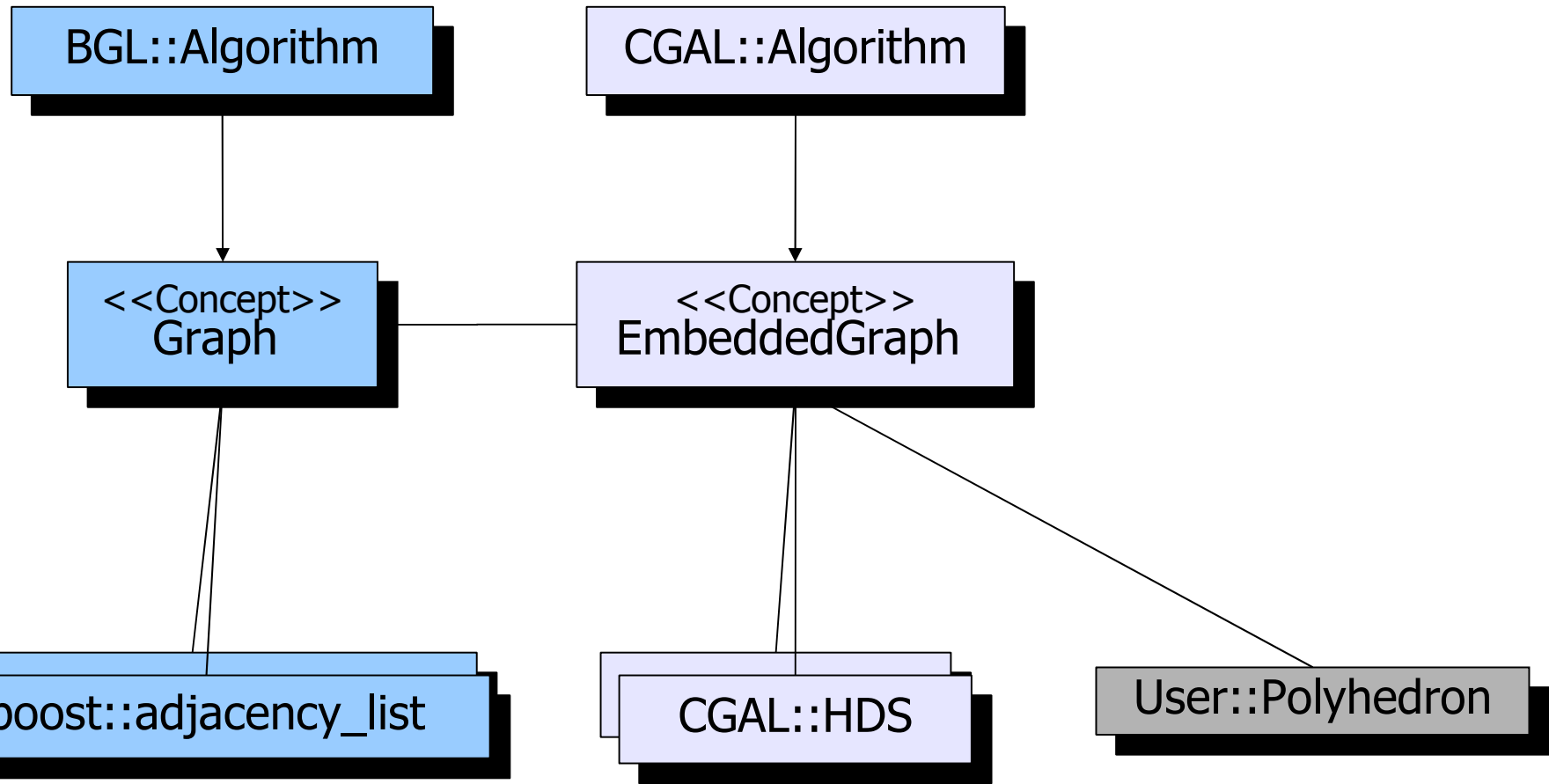


Courtesy: P.Schroeder, Caltech

From A BGL Glue Layer for CGAL



To BGL Style CGAL Algorithms



Demo CGAL + OpenMesh

CGAL::Turk_Lindstrom_surface_simplification

<<Concept>>
EmbeddedGraph

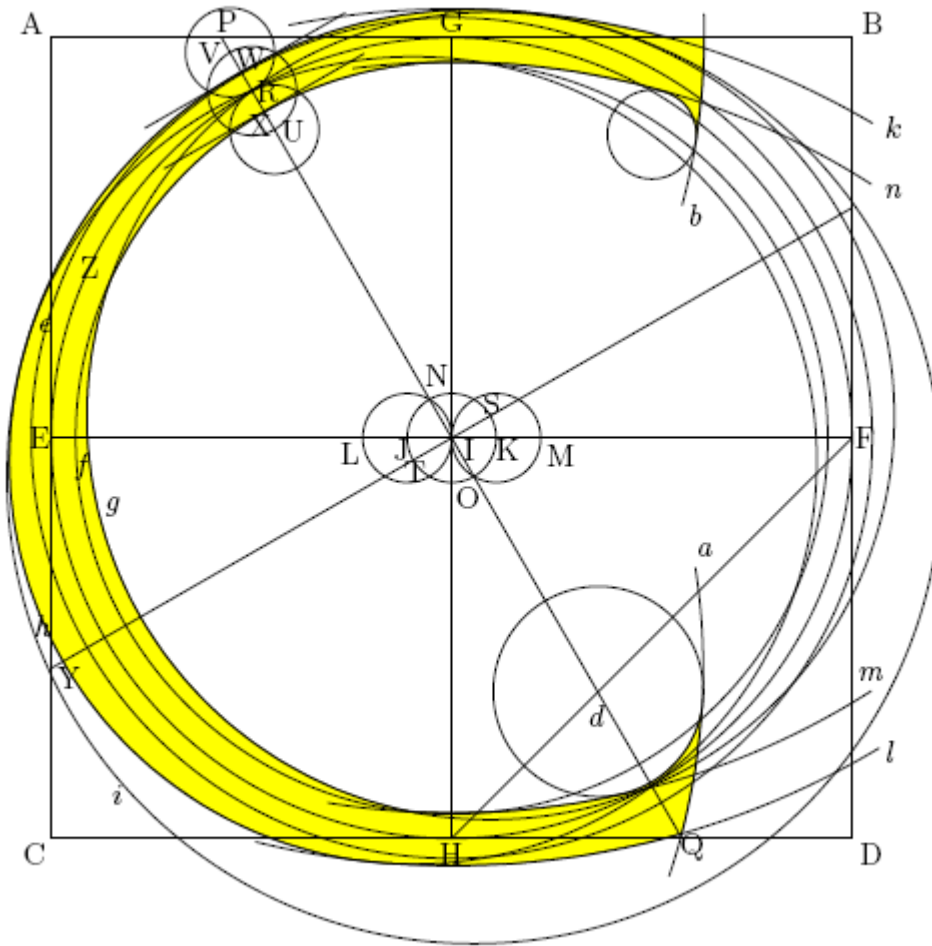
CGAL::HDS

OpenMesh::Polyhedron

```
graph TD; A[CGAL::Turk_Lindstrom_surface_simplification] --> B[<<Concept>> EmbeddedGraph]; B --> C[CGAL::HDS]; B --> D[OpenMesh::Polyhedron];
```

Summary: Overview

- Open Source project
- Clear focus on geometry
- Interfaces with de facto standards/leaders: STL, Boost, GMP, Qt, blas
- Robust and fast through exact geometric computing
- Easy to integrate through generic programming

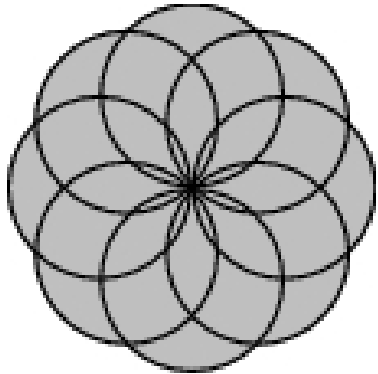


CGAL for 2D Vector Graphics

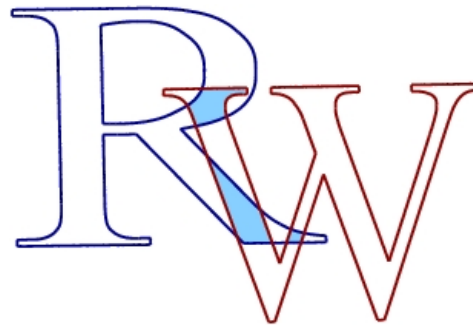
Andreas Fabri
GeometryFactory

Boolean Operations

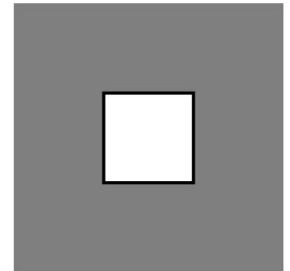
Union



Intersection



Complement



CGAL Boolean Operations can deal explicitly with

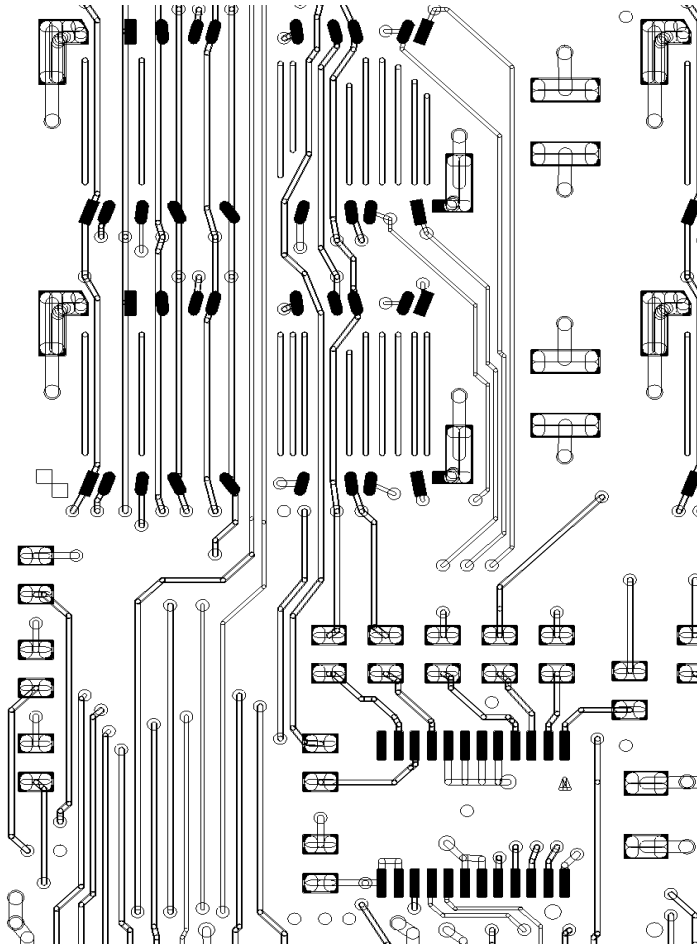
Circular arcs

Bézier Curves

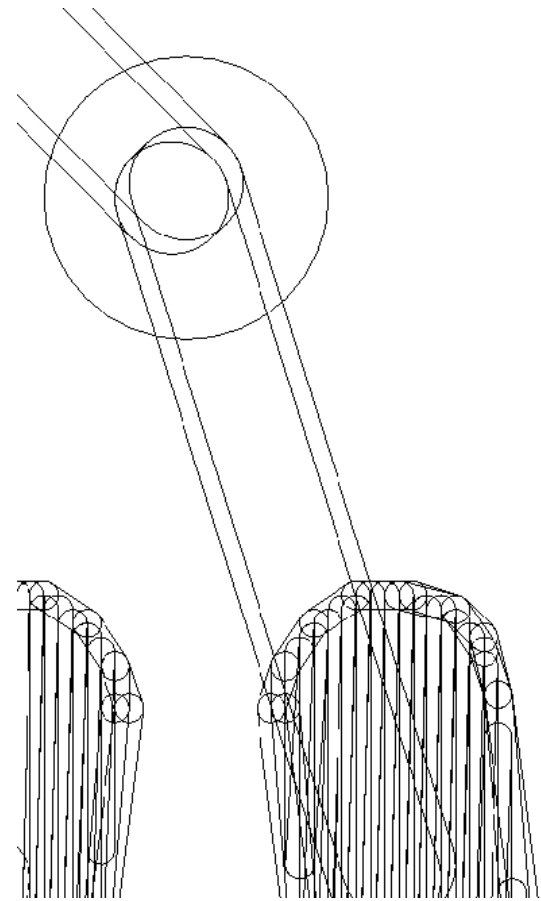
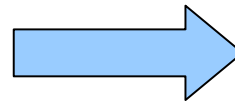
Line Segments

Boolean Ops on Circular Arcs Demo

dxf file of a printed circuit board with circular arcs

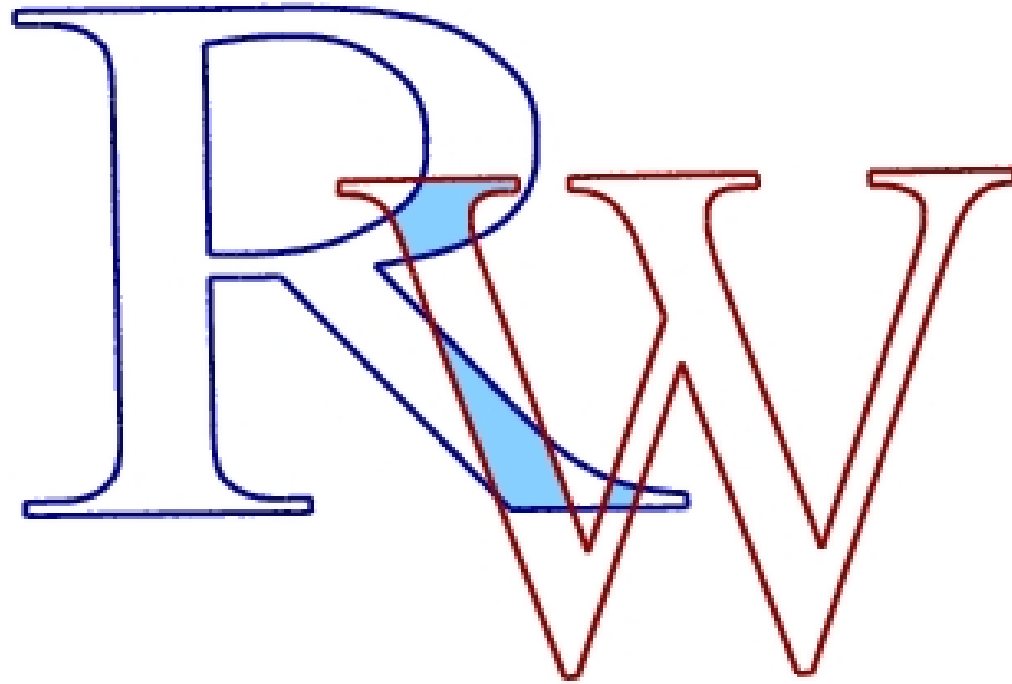


Zoom in



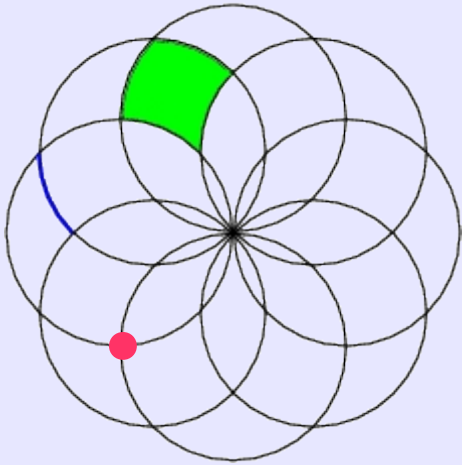
Boolean Ops on Bézier Curves

True Type fonts are Bézier curves

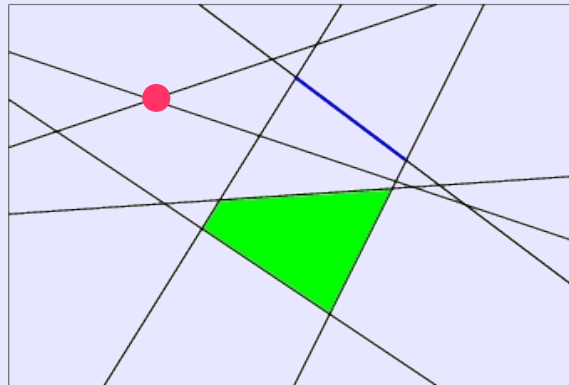


Background: 2D Arrangement

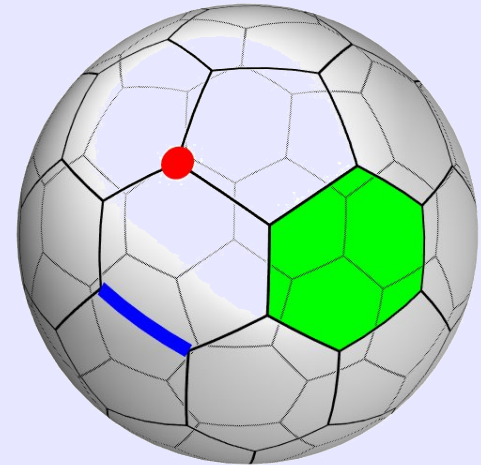
Given a collection of curves on a surface, the **arrangement** is the partition of the surface into **vertices**, **edges** and **faces** induced by the curves



An arrangement of circles in the plane



An arrangement of lines in the plane



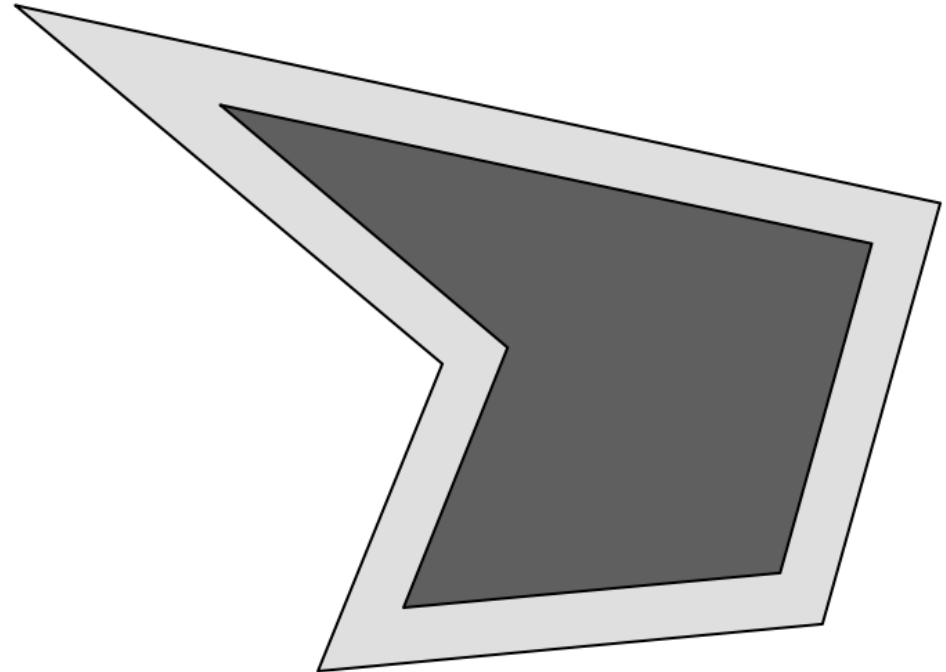
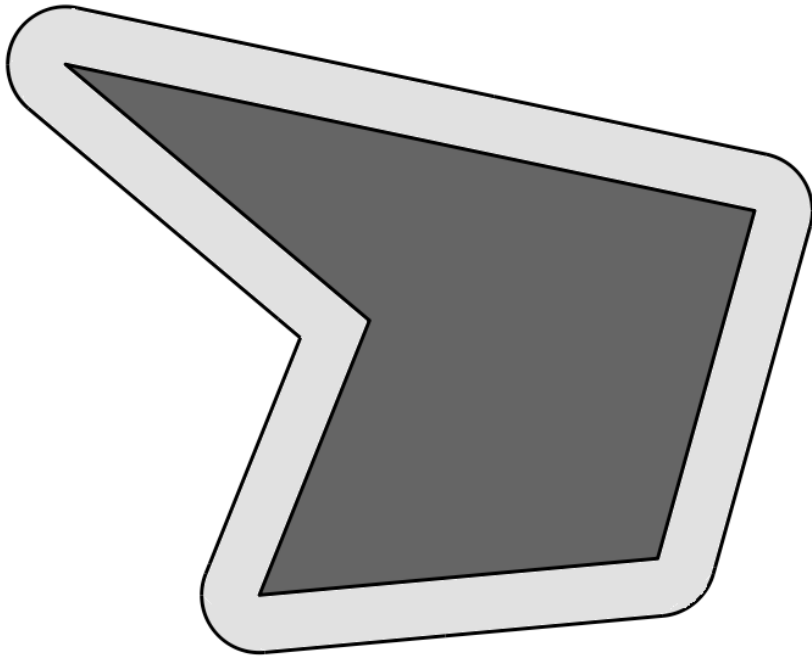
An arrangement of geodesic arcs on the sphere

Arrangement_2<Geometry>

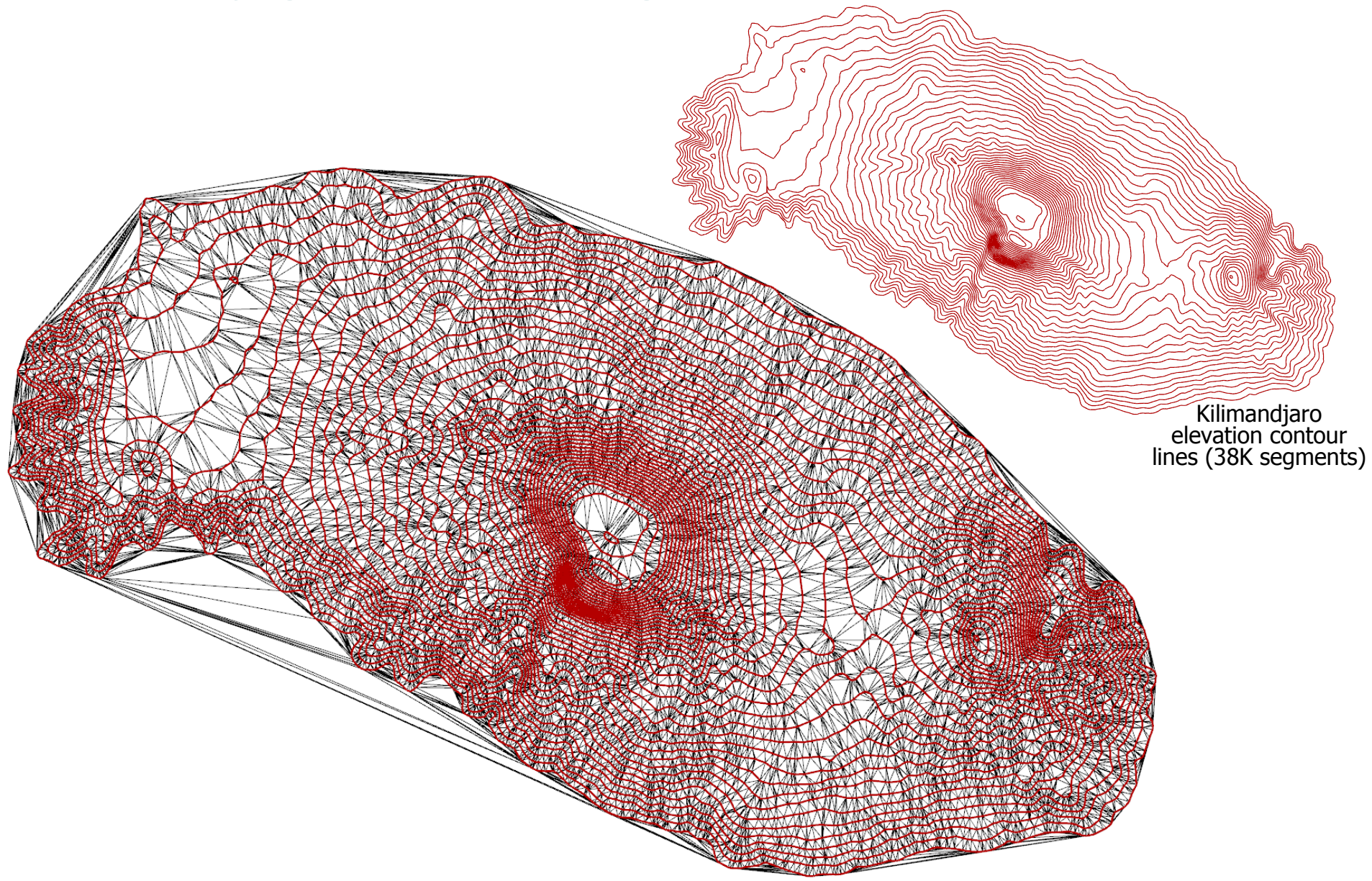
- Constructs, maintains, modifies, traverses, queries, and presents subdivisions of the plane
- Robust and exact
 - All inputs are handled correctly (including degenerate)
 - Exact number types are used to achieve exact results
- Efficient
- Generic
 - Easy to interface, extend, and adapt
 - Notification mechanism for change propagation
- Modular
 - *Geometric* and *topological* aspects are separated

Polygon Offsets

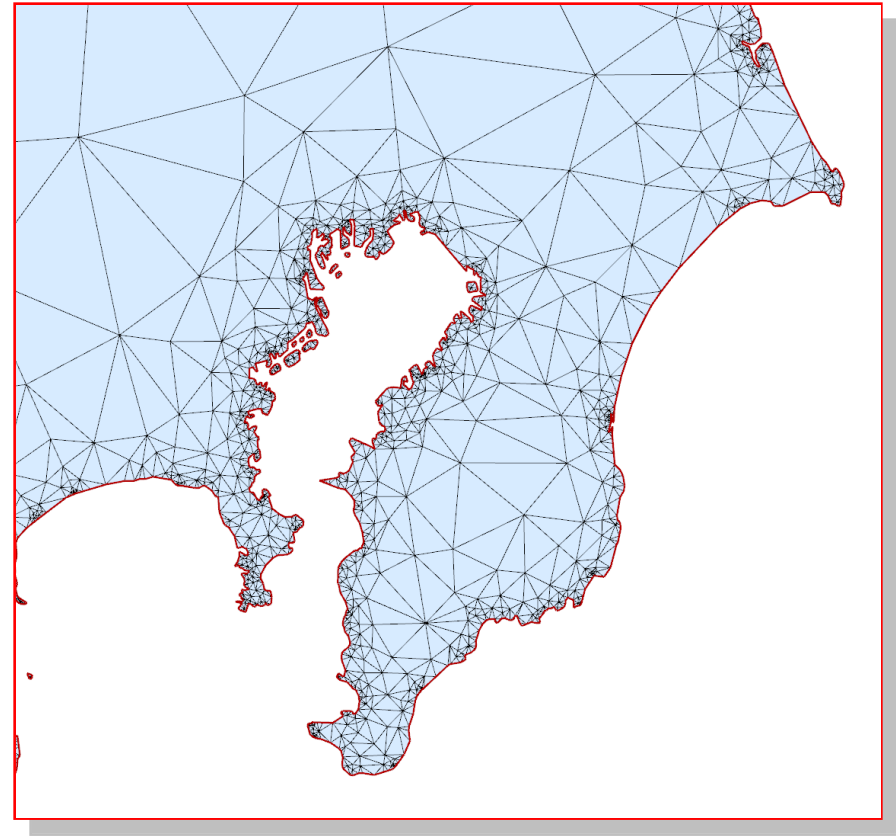
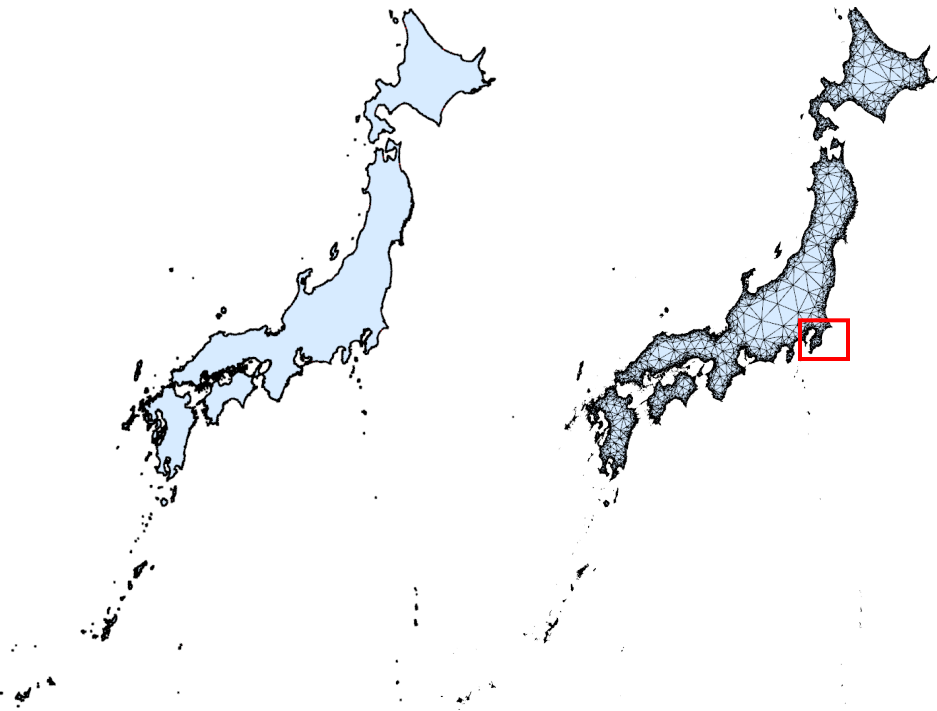
- Based on Minkowski sums, with segments and circular arcs.
- Based on straight skeleton, with segments only.



Polygon Triangulation Demo

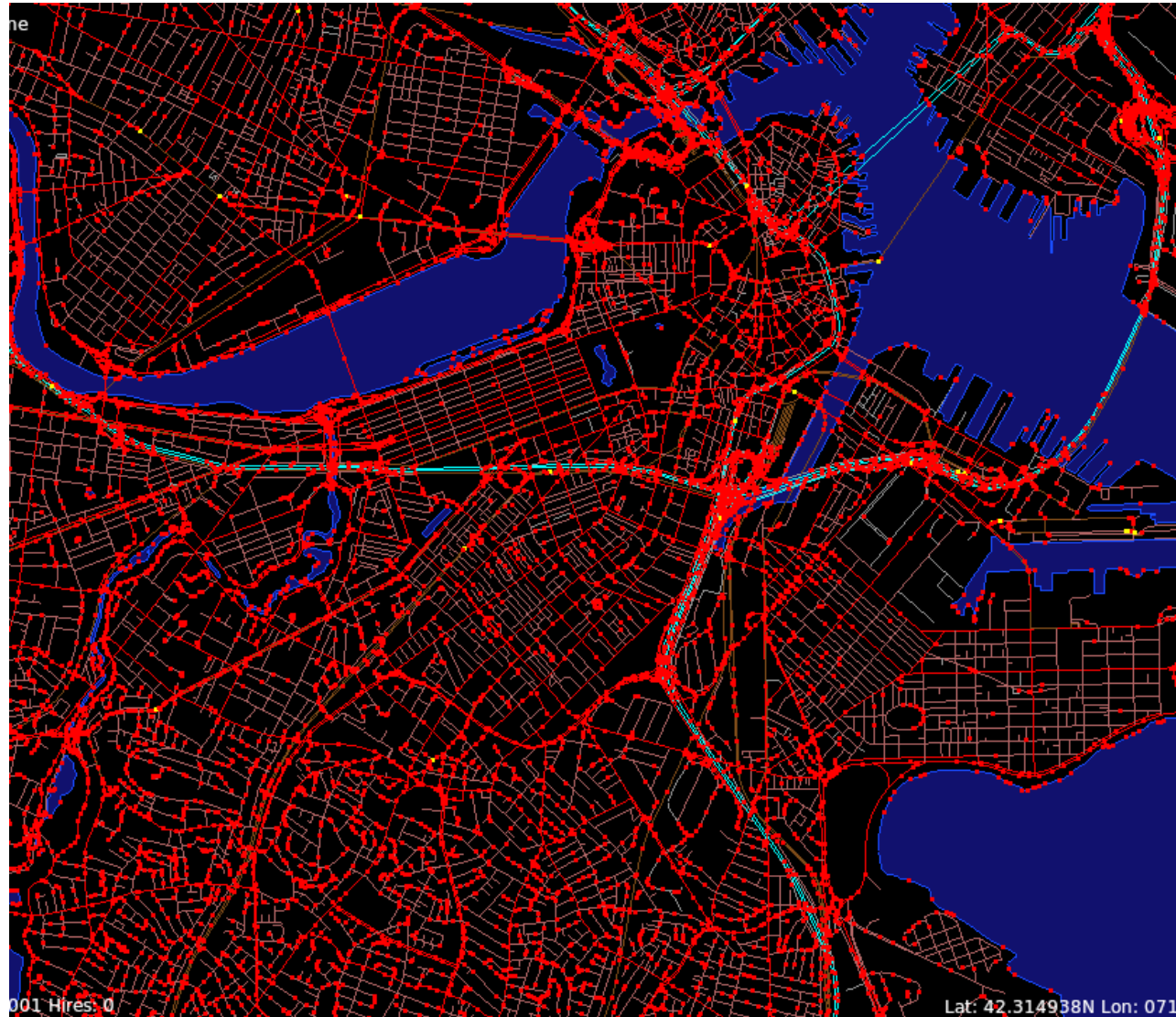


Polygon Mesh Generation Demo



Simultaneous Polyline Simplification

Red points were removed



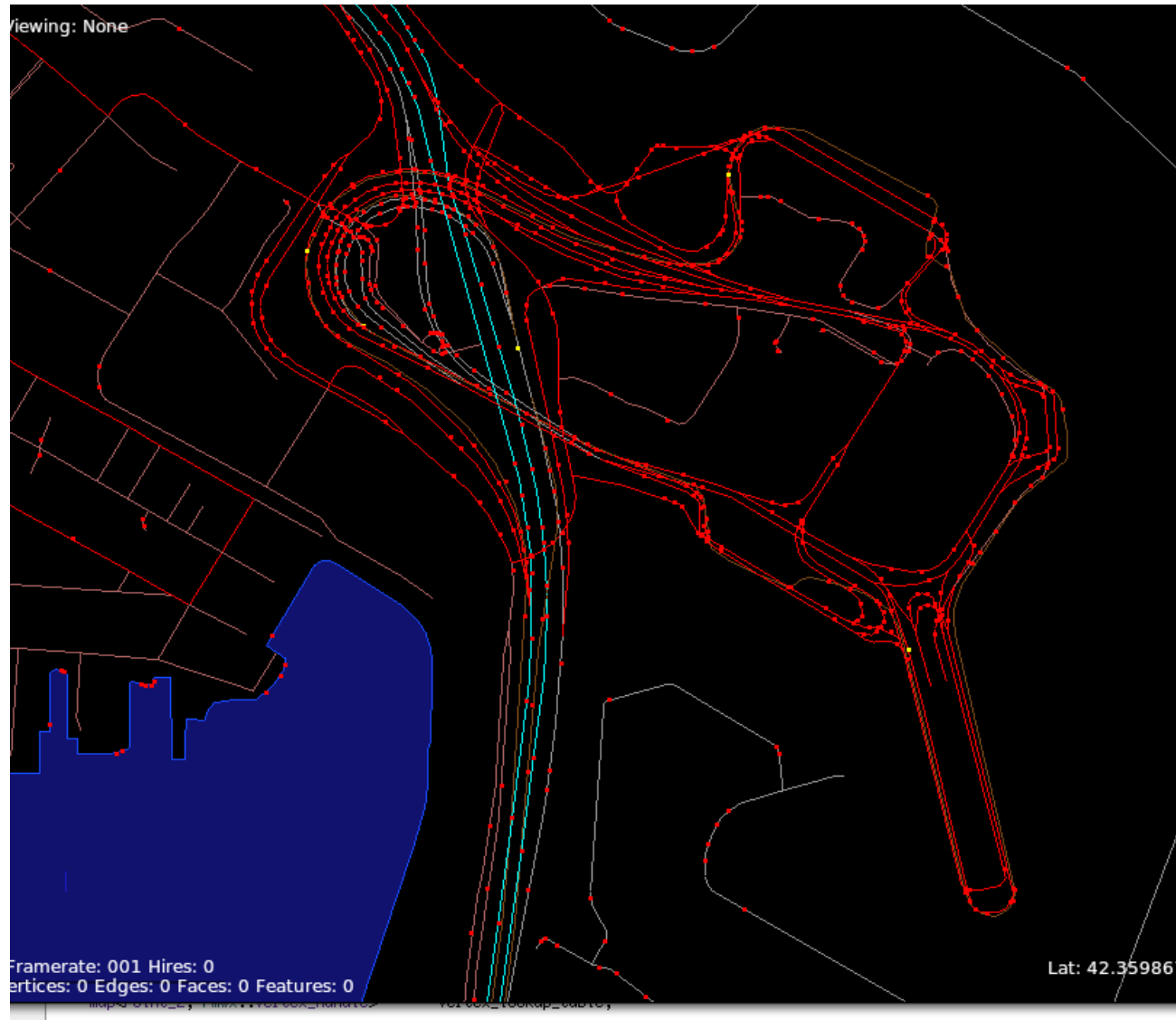
Courtesy: Laminar Research

Simultaneous Polyline Simplification

Input is the
transportation and
water layers of
OpenStreetMap

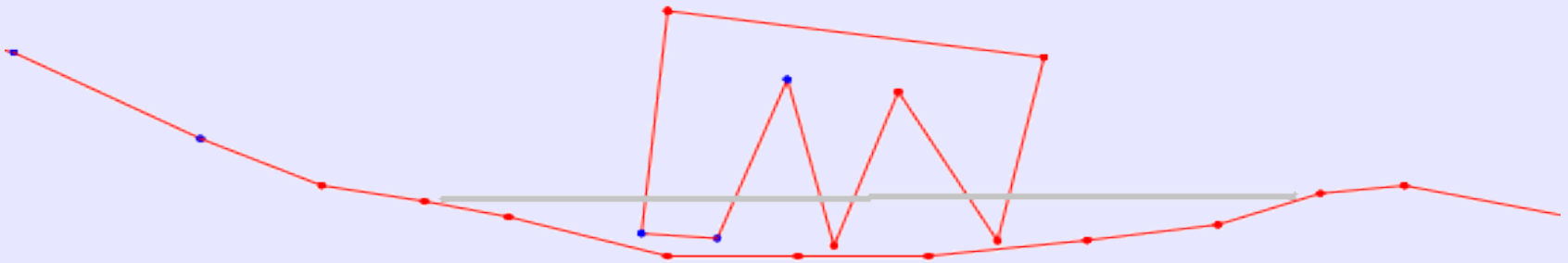
Red points were
removed

Courtesy: Laminar Research



Simultaneous Polyline Simplification

- Implementation of [Dyken et al]
- Based on CGAL::Constrained_delaunay_2
- Guarantees that after simplification
 - islands stay islands
 - isolines do not intersect



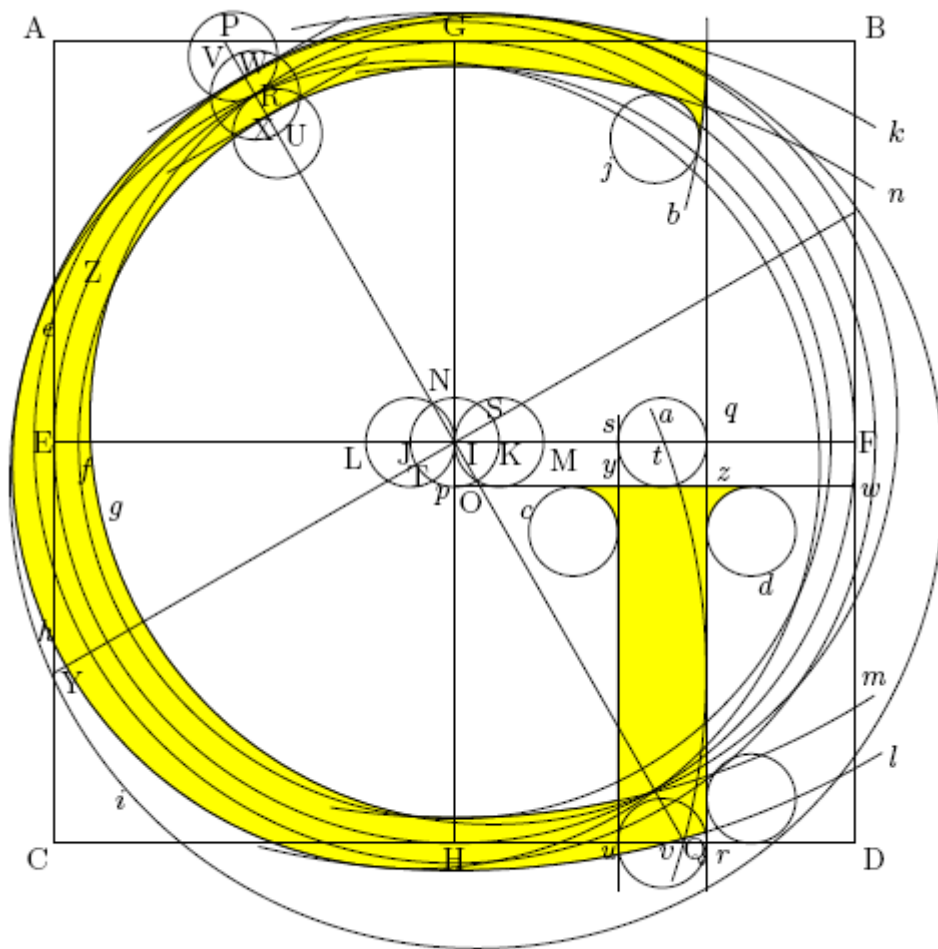
Vector Graphics on the Sphere

- Arrangement_2<Geometry, **Embedding**>
- Boolean operations
- Map overlay
- Voronoi diagram
- Point location
- Convex decomposition



Summary: CGAL for Vector Graphics

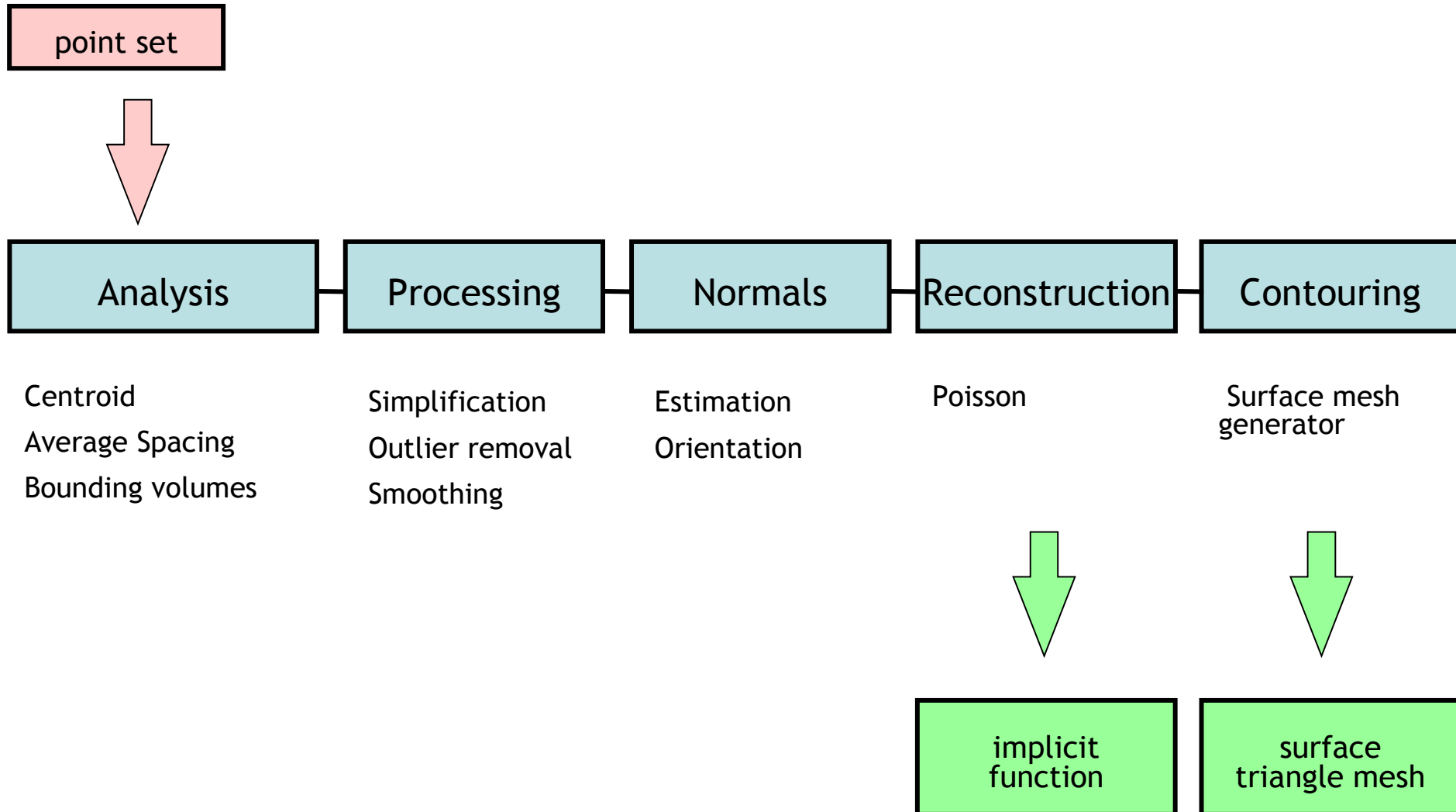
- Rich collection of 2D geometric algorithms
- Modular and generic design
- Linear and curved primitives
- Useful in many application domains



CGAL for Point Sets

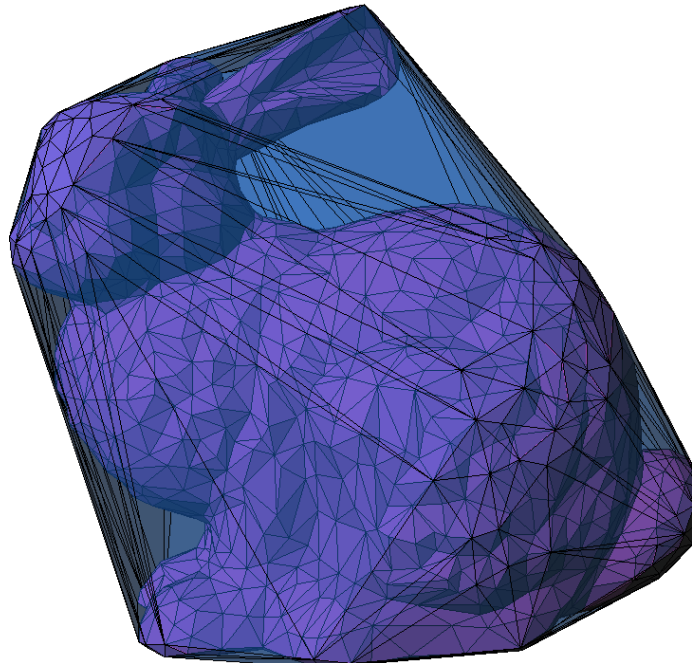
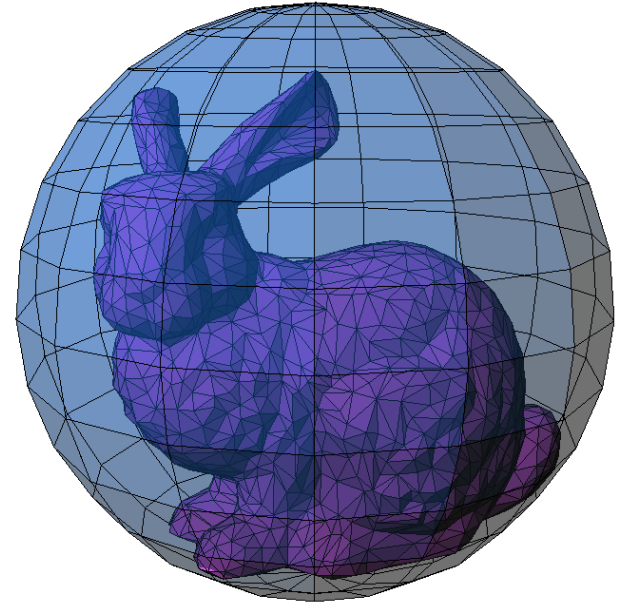
Pierre Alliez
INRIA

Surface Reconstruction Pipeline



Bounding Volumes

- Convex hull
- Bounding sphere



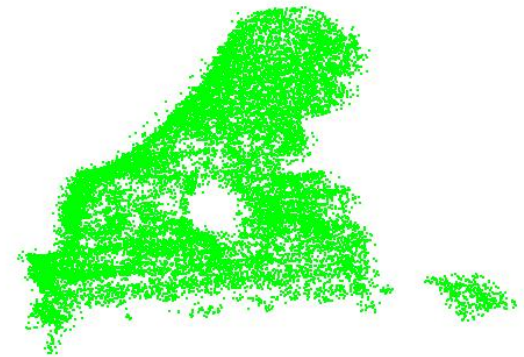
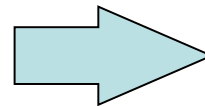
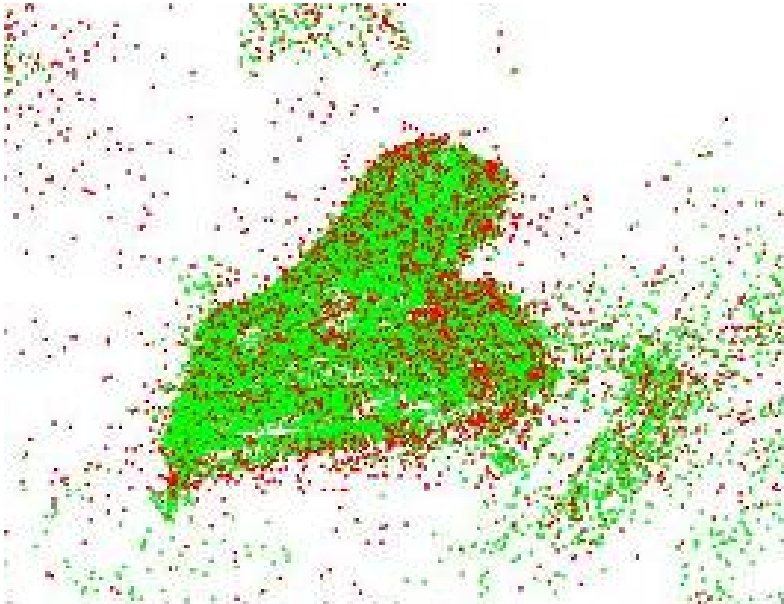
Principal Component Analysis

Linear least squares
fitting on sets of 3D
points



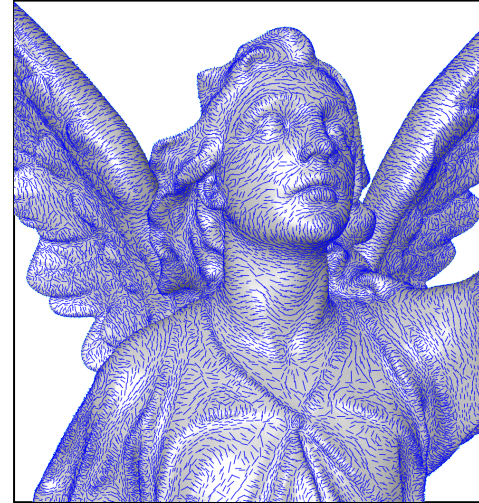
Outlier Removal

- Sort w.r.t. sum of squared distances to k-nearest neighbors (CGAL::K_nearest_neighbor_search) and cut at specified percentile.



Estimation of Curvatures

- Estimates general differential properties (Monge form) on point sets.
- Through polynomial (d-jet) fitting



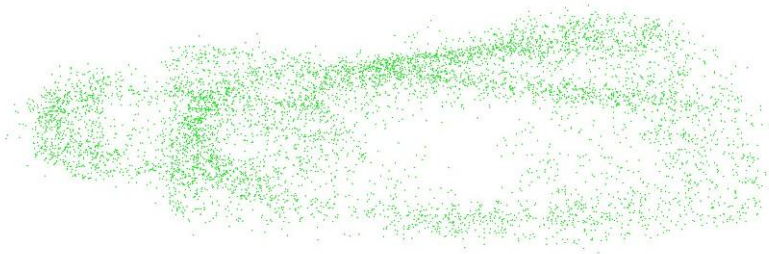
min curvature directions



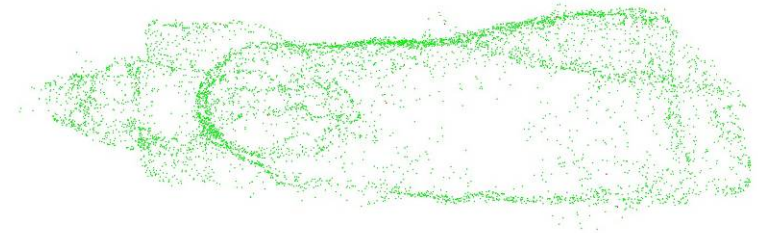
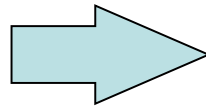
max curvature directions

Point Cloud Smoothing

- For each point
 - Find k-nearest neighbors
 - fit jet (smooth parametric surface)
 - project onto jet



(noisy point set)



(smoothed point set)

Surface Reconstruction

Poisson Surface Reconstruction

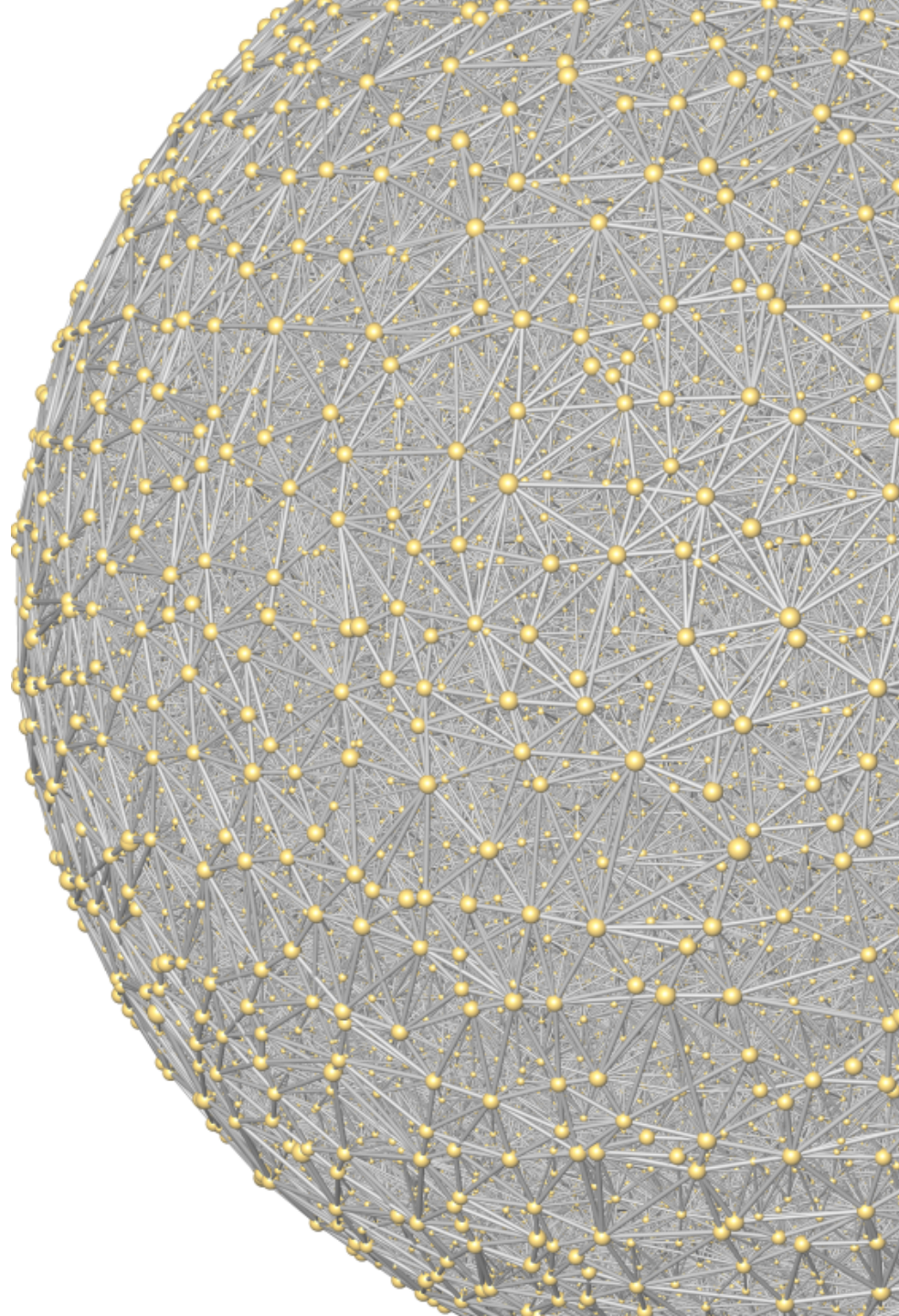
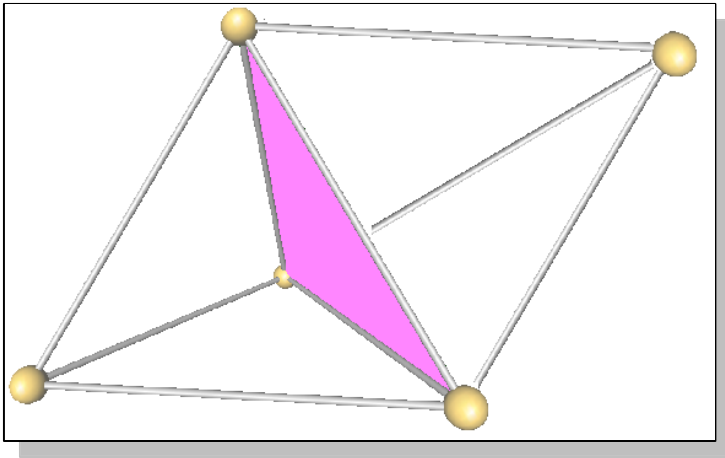
[Kazhdan-Bolitho-Hoppe, SGP 2006]

- Solves for an implicit function (~indicator function)
- Isosurface extracted by `CGAL::Surface_mesher`



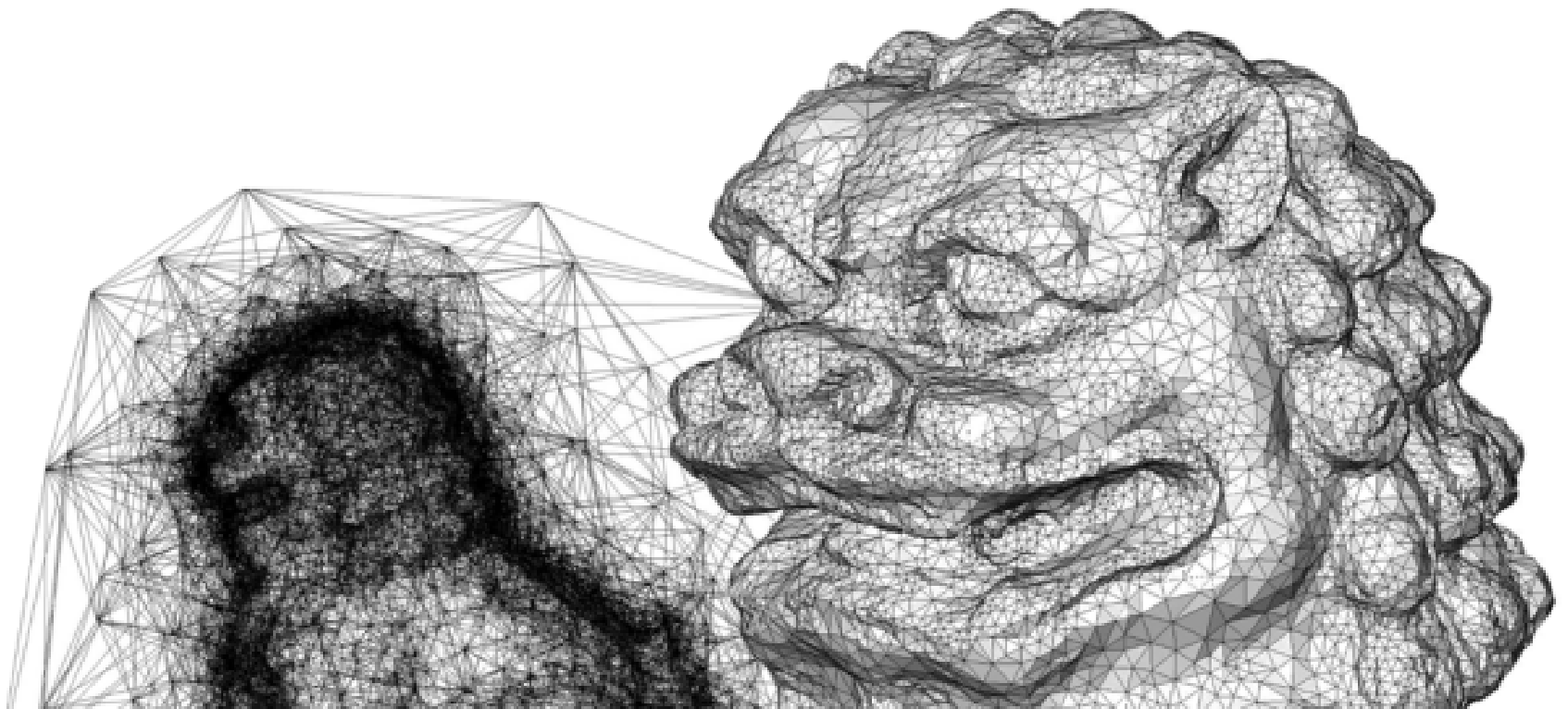
3D Triangulations

- Delaunay
- Fully dynamic
- 1M 3D points in 16 sec



Surface Reconstruction Demo

- Solves for the Poisson equation onto the vertices of a (refined) 3D Delaunay triangulation.



Summary: CGAL for Point Sets

- Algorithms are modular components
 - in this course: positioned along the surface reconstruction pipeline.
 - can be used individually
- Poisson reconstruction is the first algorithm of the surface reconstruction package.

CGAL for Modeling and Processing of Polyhedral Surfaces

Andreas Fabri

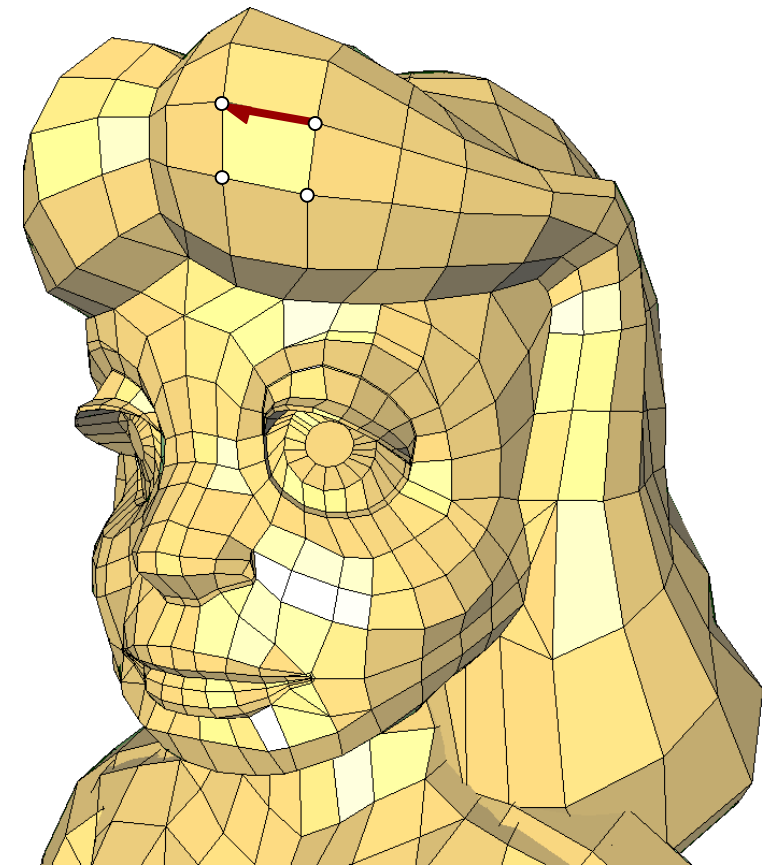
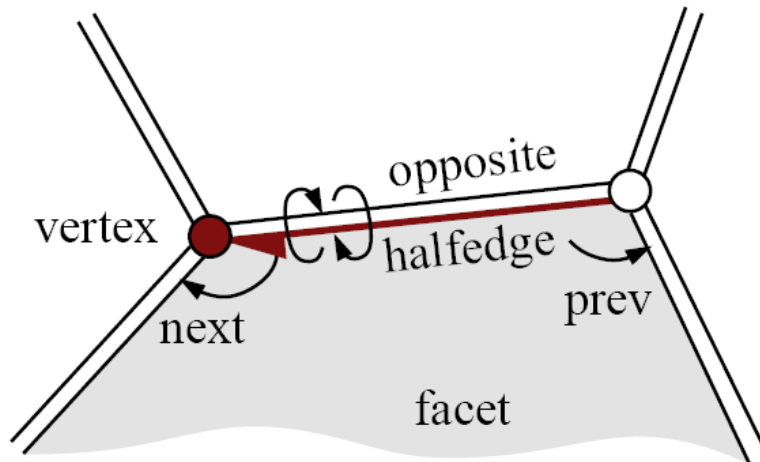
GeometryFactory

Outline

- Polyhedral Surface
 - Halfedge data structure
 - Euler Operators
 - Customization
- Algorithms for Geometric Modelling and Geometry Processing

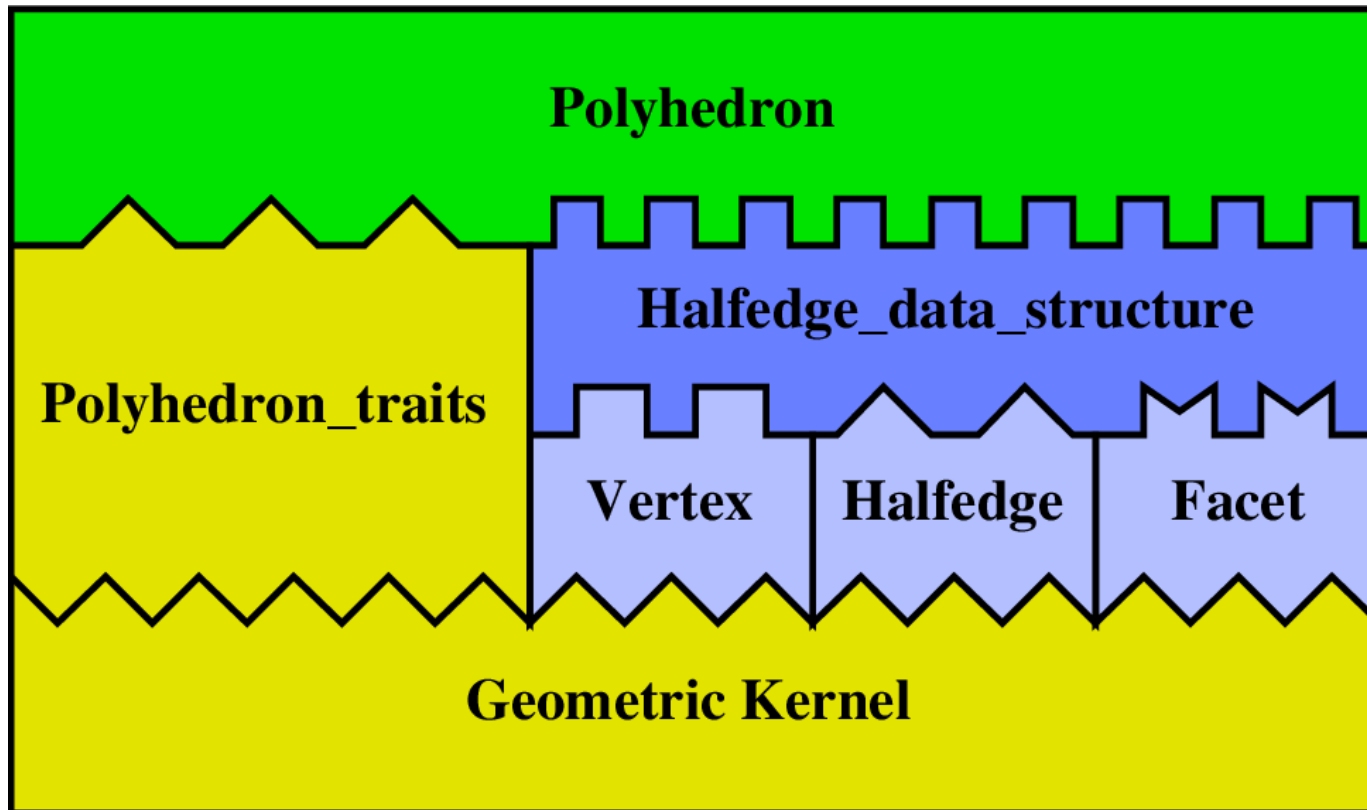
Halfedge Data Structure

Represented by vertices, edges, facets and an **incidence relation** on them, restricted to orientable 2-manifolds with boundary.



Polyhedron

Building blocks assembled with C++ templates

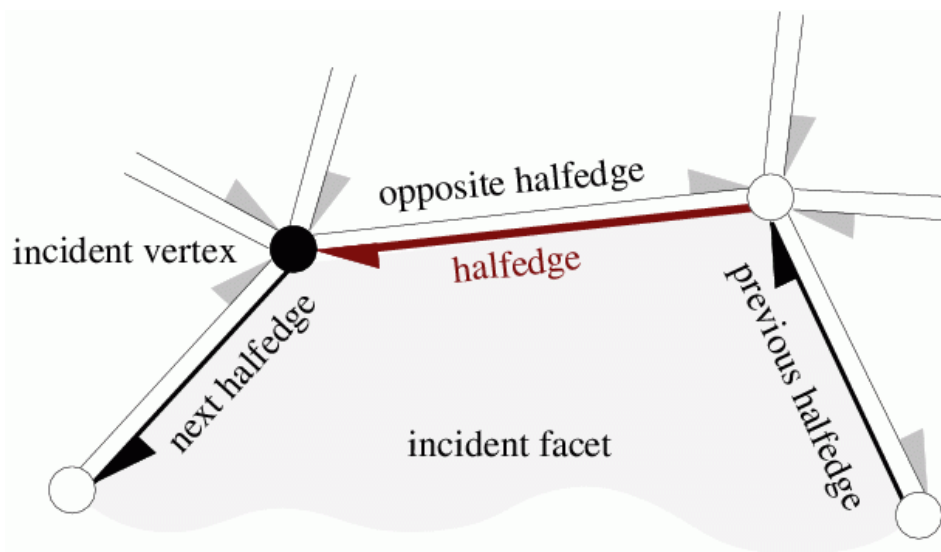


Default Polyhedron

Vertex	
Halfedge_handle	halfedge()
Point&	point()
.....	...

Halfedge	
Halfedge_handle	opposite()
Halfedge_handle	next()
Halfedge_handle	prev()
Vertex_handle	vertex()
Facet_handle	facet()
.....	...

Facet	
Halfedge_handle	halfedge()
Plane&	plane()
Normal&	normal()
Color&	color()
.....	...

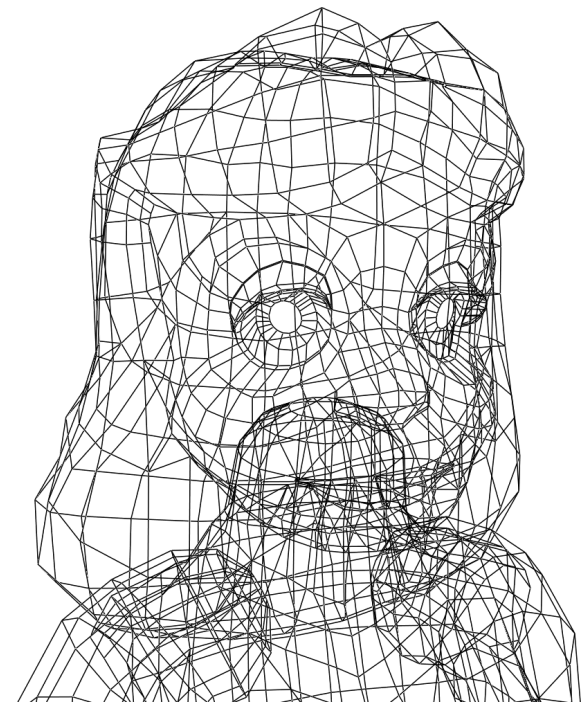
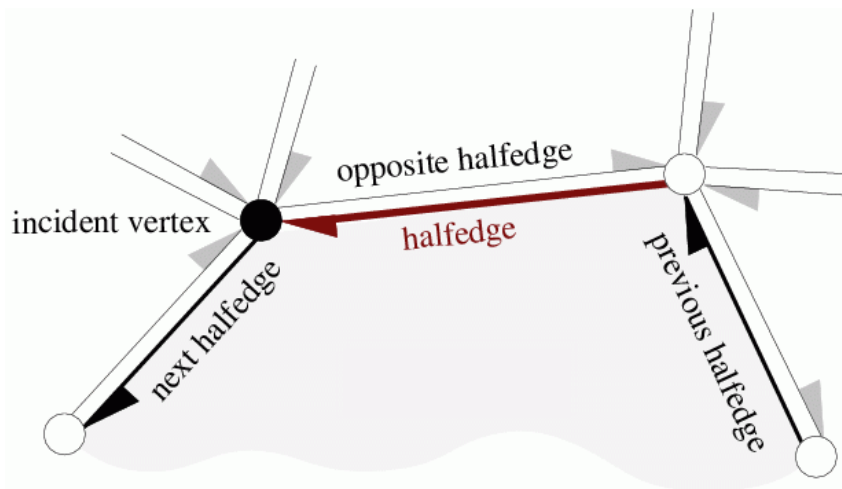


Flexible Data Structure

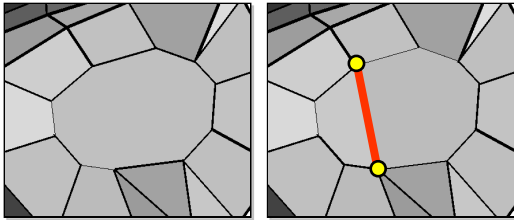
Vertex	
Halfedge_handle	halfedge()
Point&	point()
.....	...

Halfedge	
Halfedge_handle	opposite()
Halfedge_handle	next()
Halfedge_handle	prev()
Vertex_handle	vertex()
Facet_handle	facet()
.....	...

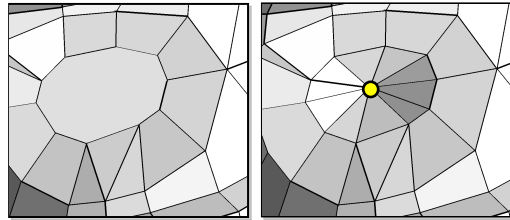
Facet	
Halfedge_handle	halfedge()
Plane&	plane()
Normal&	normal()
Color&	color()
.....	...



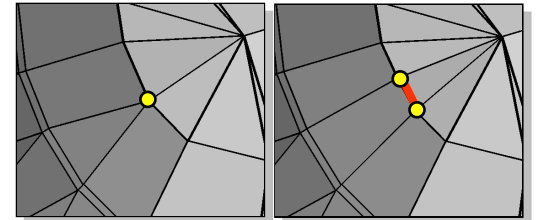
Euler Operators



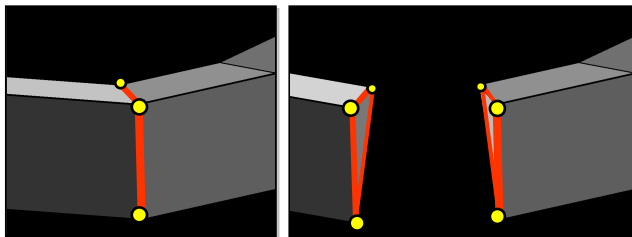
split_facet
join_facet



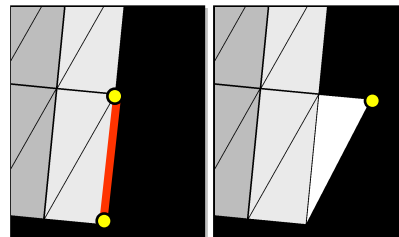
create_center_vertex
erase_center_vertex



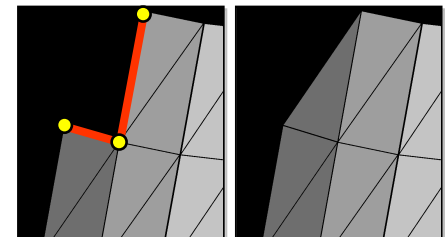
split_vertex
join_vertex
(aka edge collapse)



split_loop
join_loop



add_vertex_and_facet
_to_border
erase_facet



add_facet_to_border
erase_facet

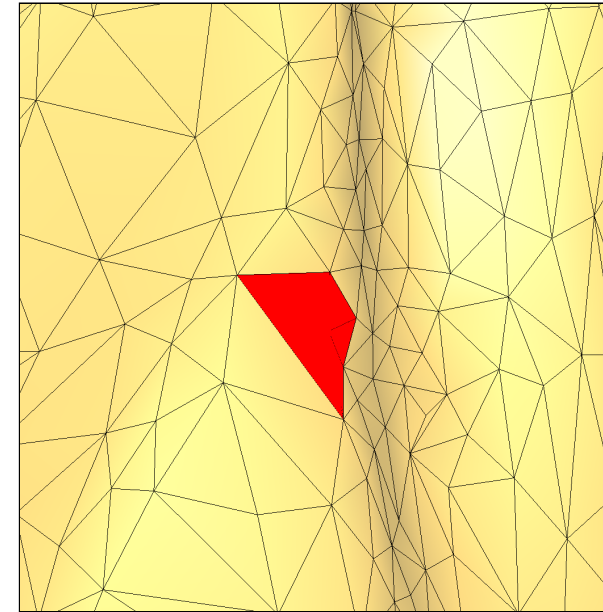
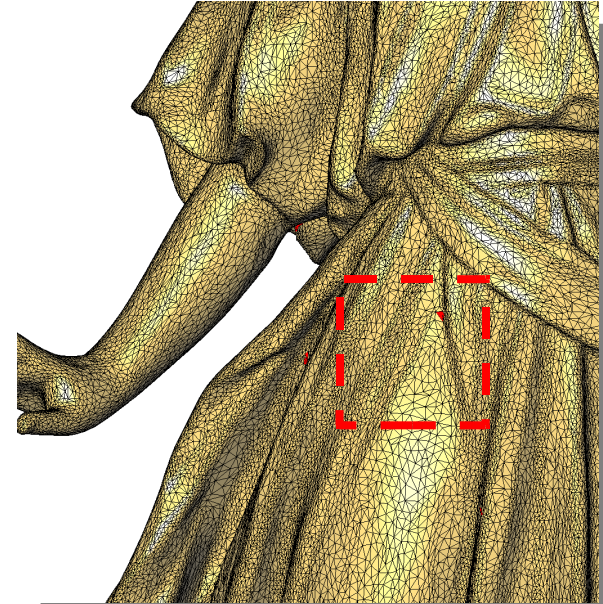
Algorithms

Algorithms

- Intersection detection
- AABB Tree
- Bounding volumes
- Boolean operations
- Kernel
- Parameterization
- Subdivision
- Principal component analysis
- Extraction of ridges
- Simplification

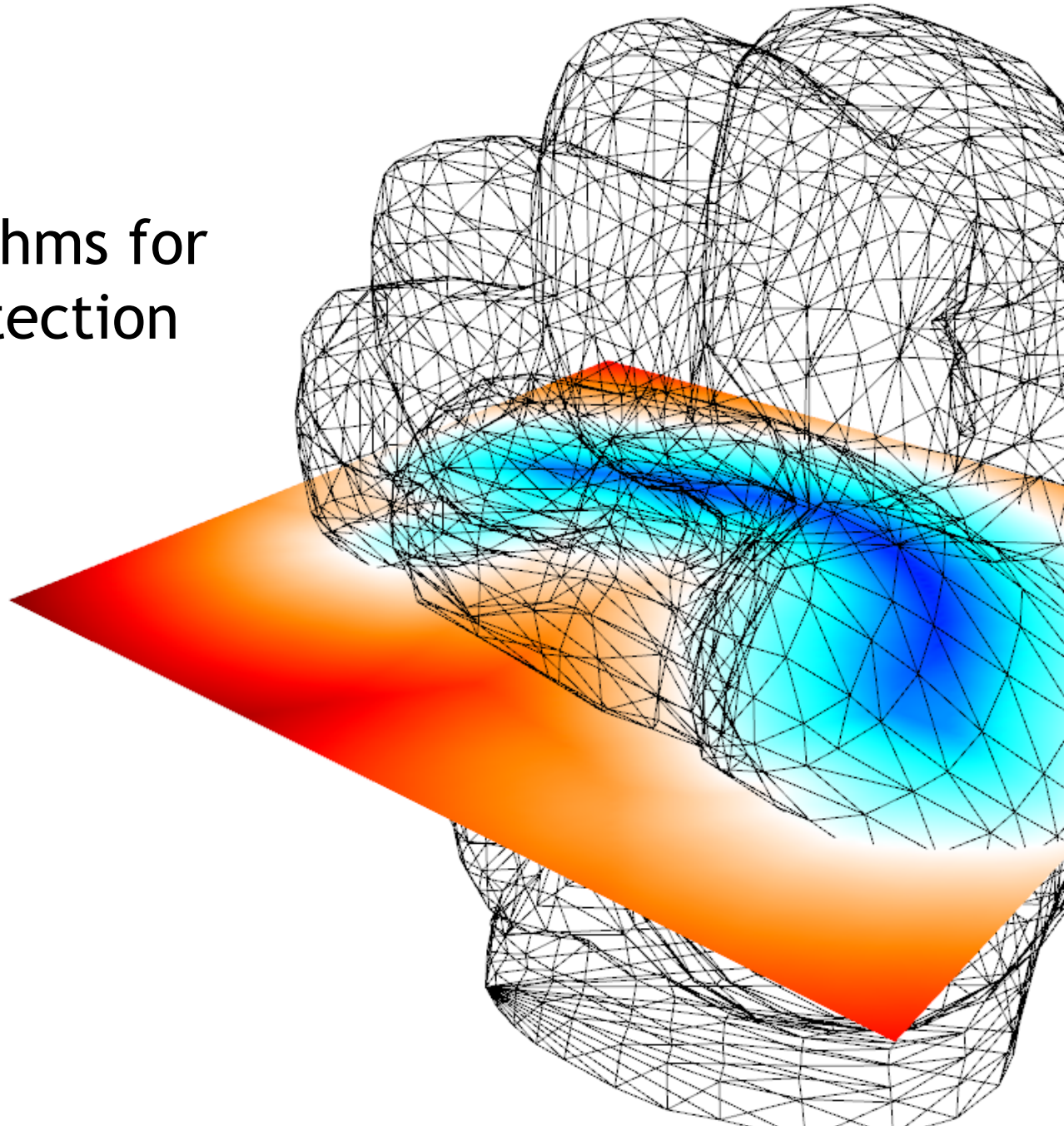
Intersection Detection

- Efficient algorithm for finding all intersecting pairs for large numbers of axis-aligned bounding boxes.
- Generic programming: Boxes can contain objects of any type



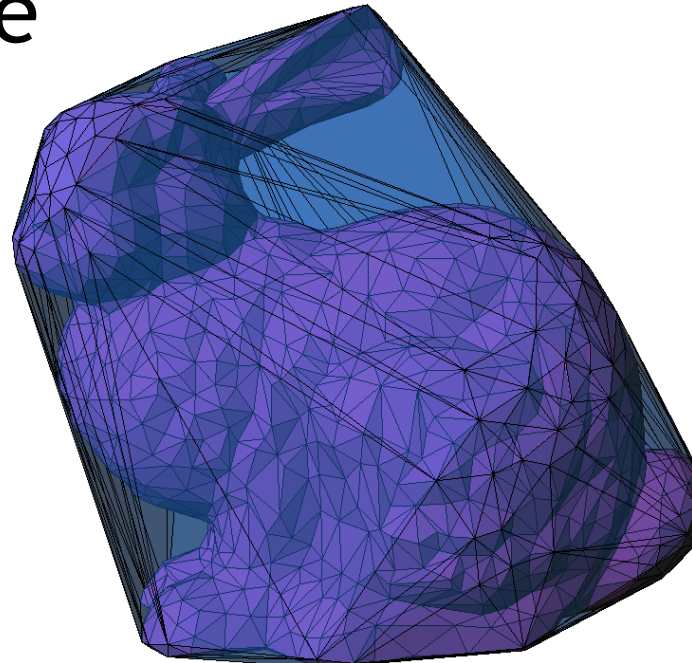
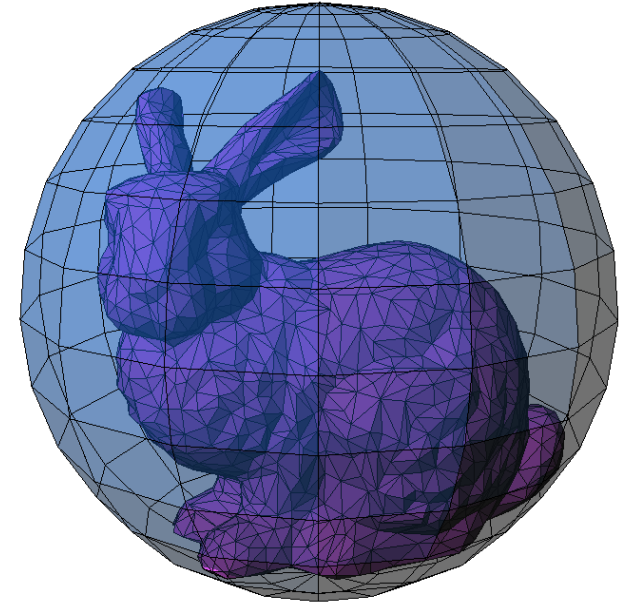
AABB Tree

Efficient algorithms for
intersection detection
and distance
computation



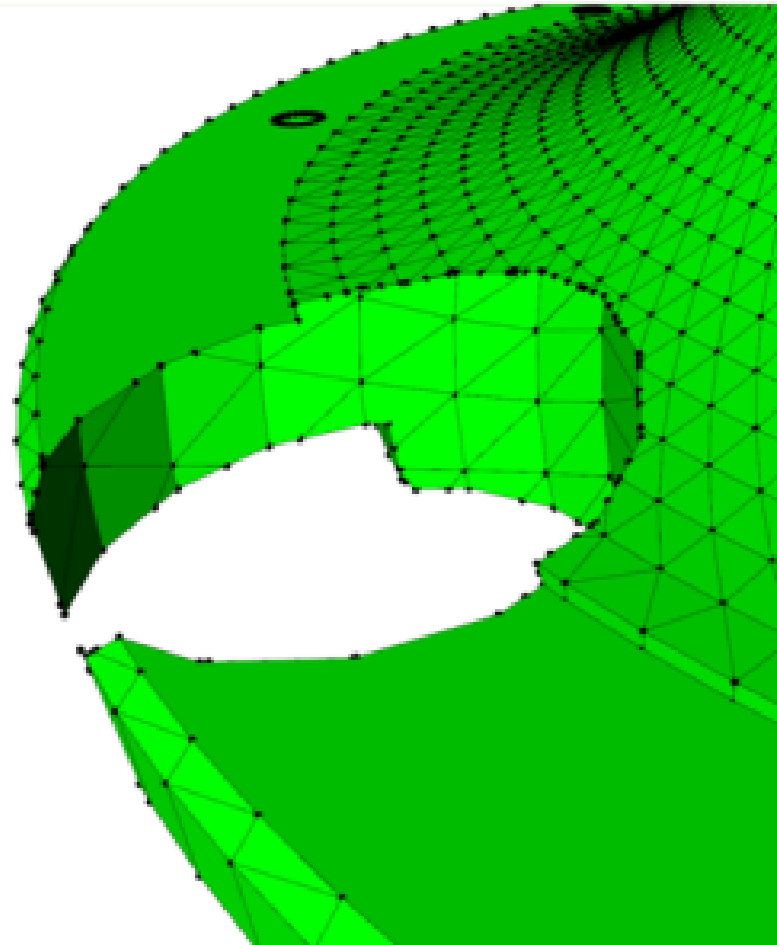
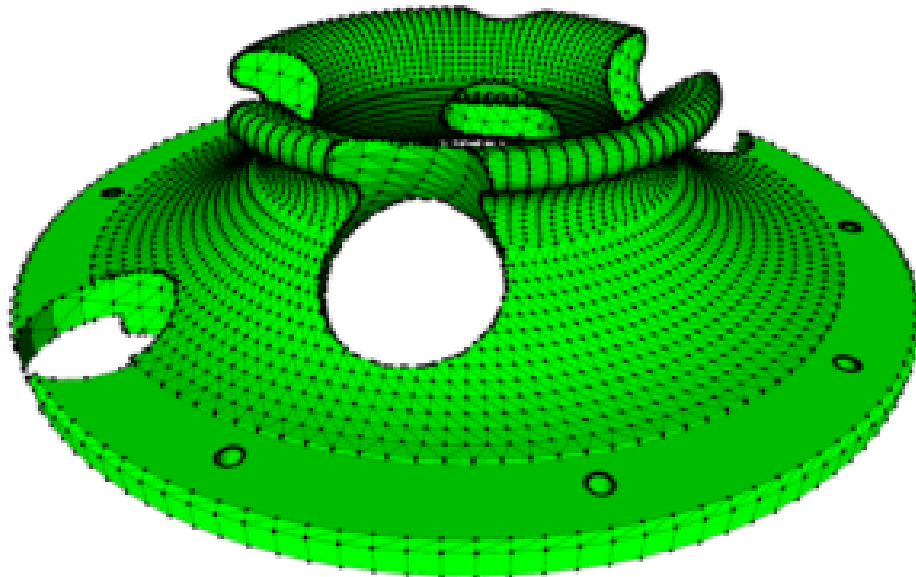
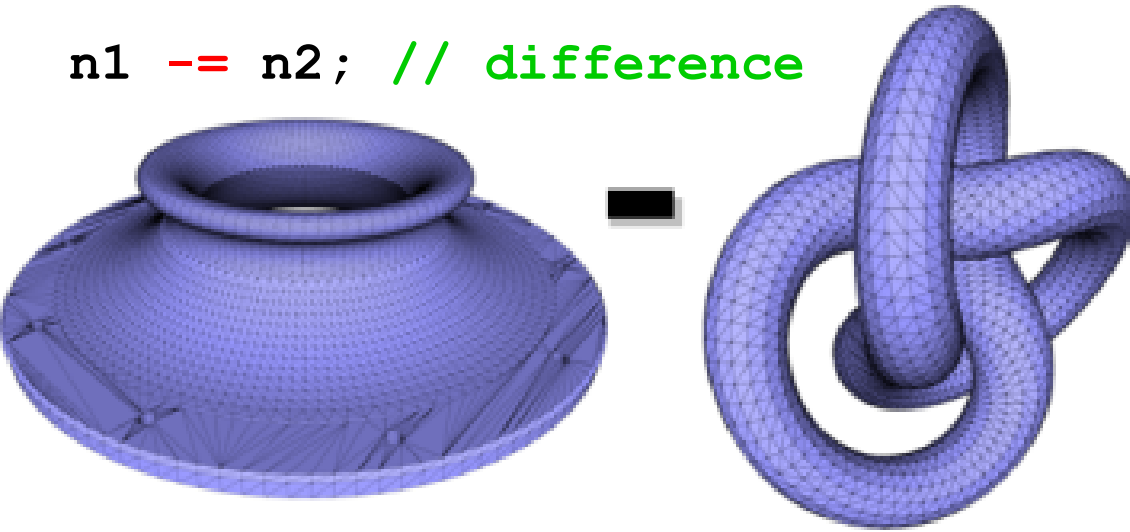
Bounding Volumes

- Convex hull
- Bounding sphere
- Bounding sphere of spheres



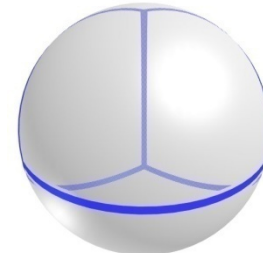
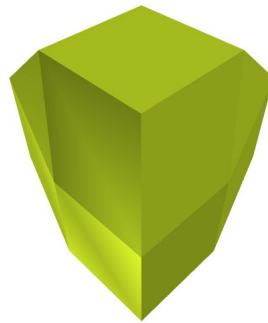
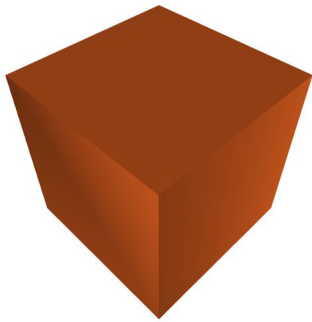
3D Boolean Operations

```
n1 -= n2; // difference
```



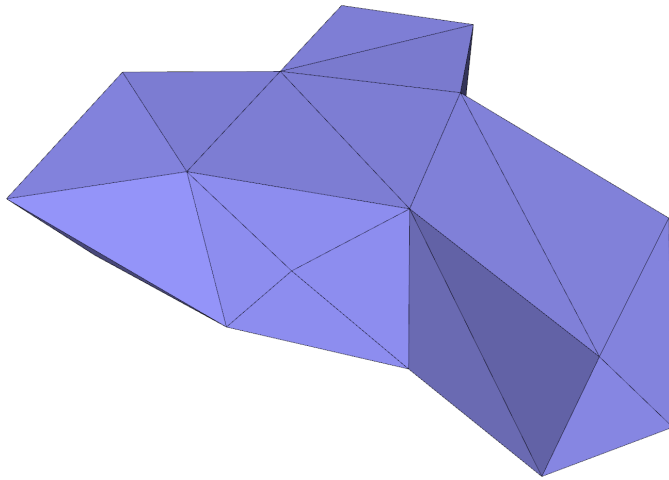
Minkowski-Sums of Polytopes

- The Gaussian map of a polytope is the decomposition of S^2 into maximal connected regions so that the extremal point is the same for all directions within one region
- The overlay of the Gaussian maps of two polytopes P and Q is the Gaussian map of the Minkowski sum of P and Q

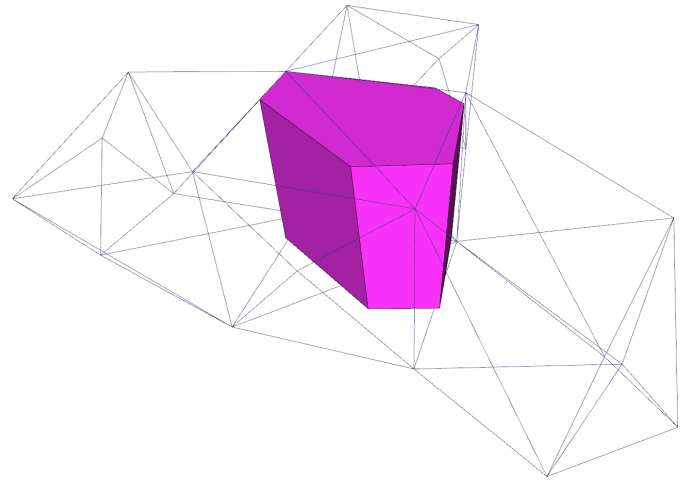


Kernel of a Polyhedron

- Intersection of all its interior half-spaces
- Uses linear programming: `CGAL::QP_solver`



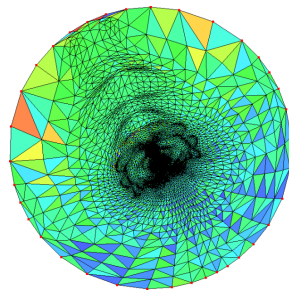
input polyhedron



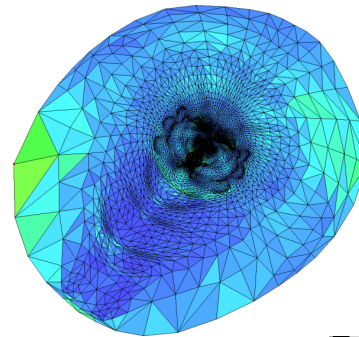
kernel

Parameterization

- Planar
- Conformal [Eck et al., Levy et al., Desbrun et al.]
- Mean value coordinates [Floater]
- ...



Fixed
boundary

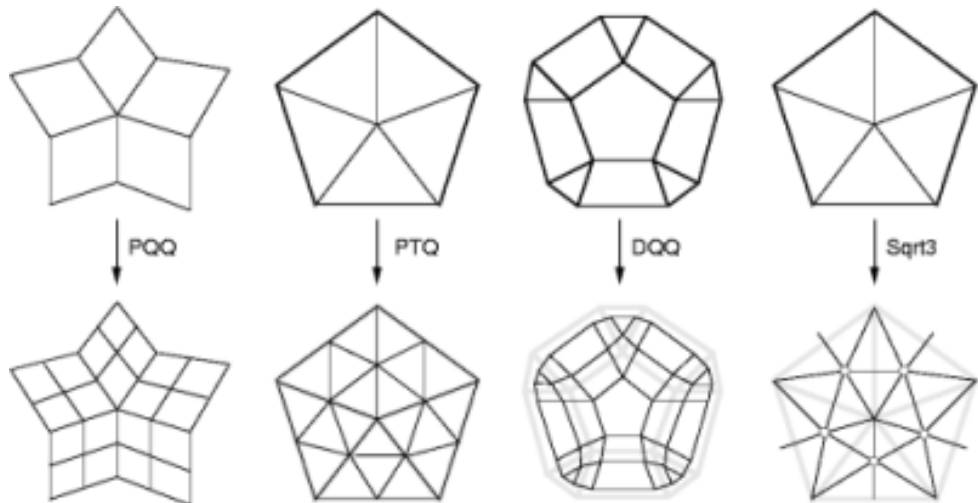
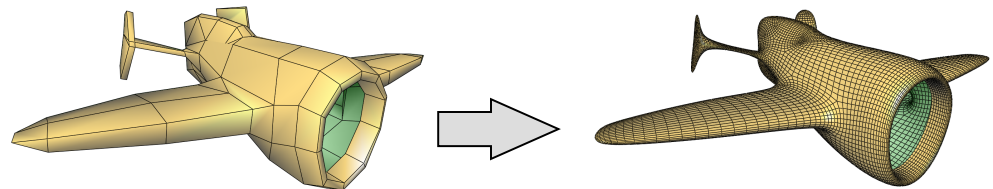


Free
boundary



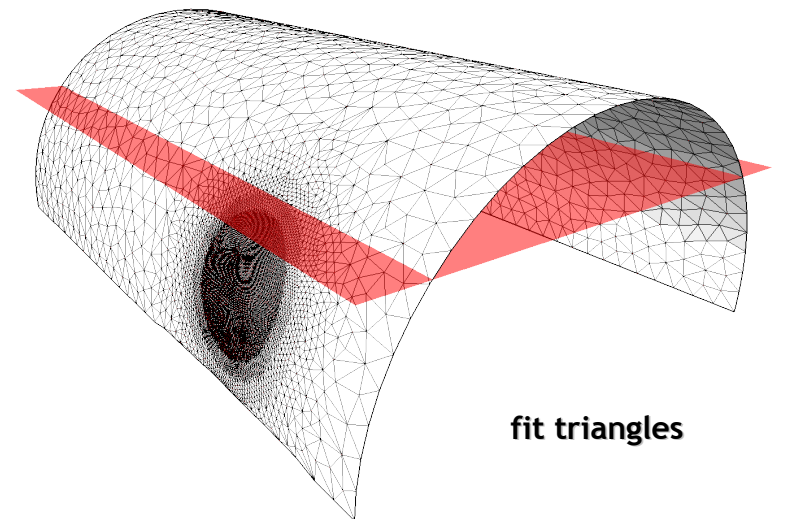
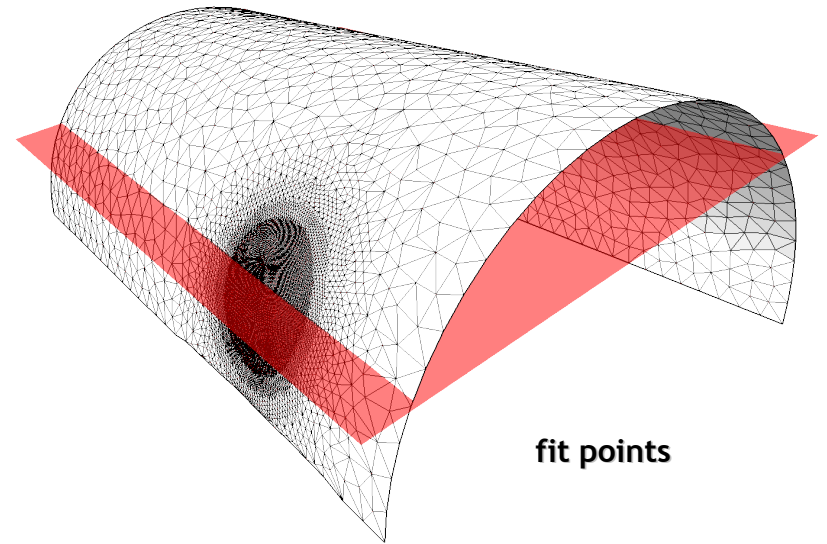
Subdivision

- Designed to work on CGAL polyhedron
- Catmull-Clark
- Loop
- Doo-Sabin
- Sqrt3
- ...



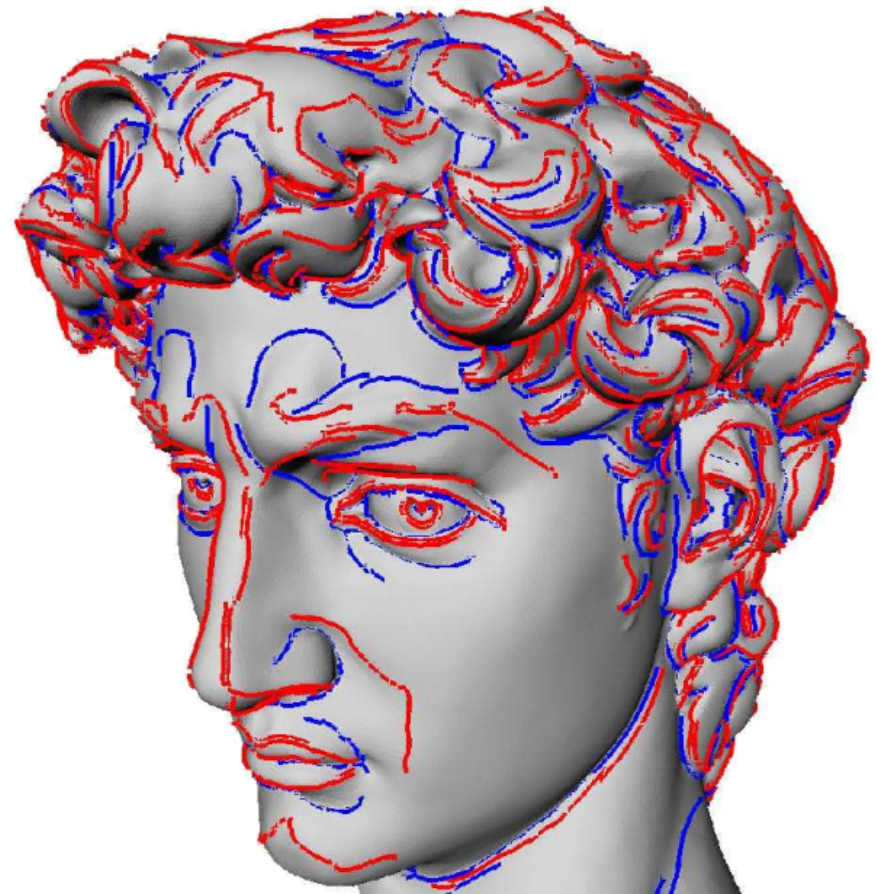
Principal Component Analysis

- Linear least squares fitting on sets of 3D points or triangles



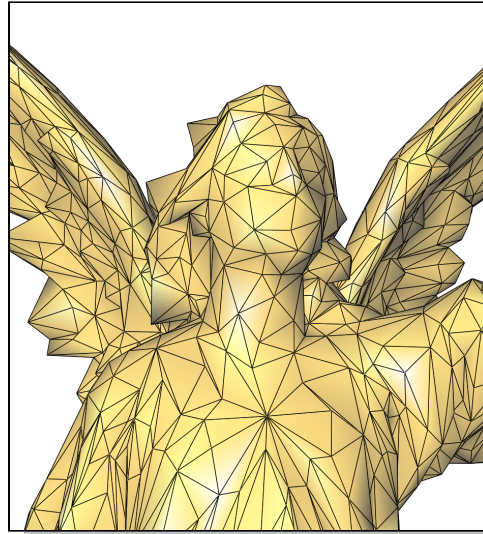
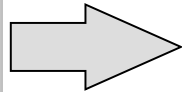
Extraction of Ridges

- Ridge: curve along which one of the principal curvatures has an extremum along its curvature line.



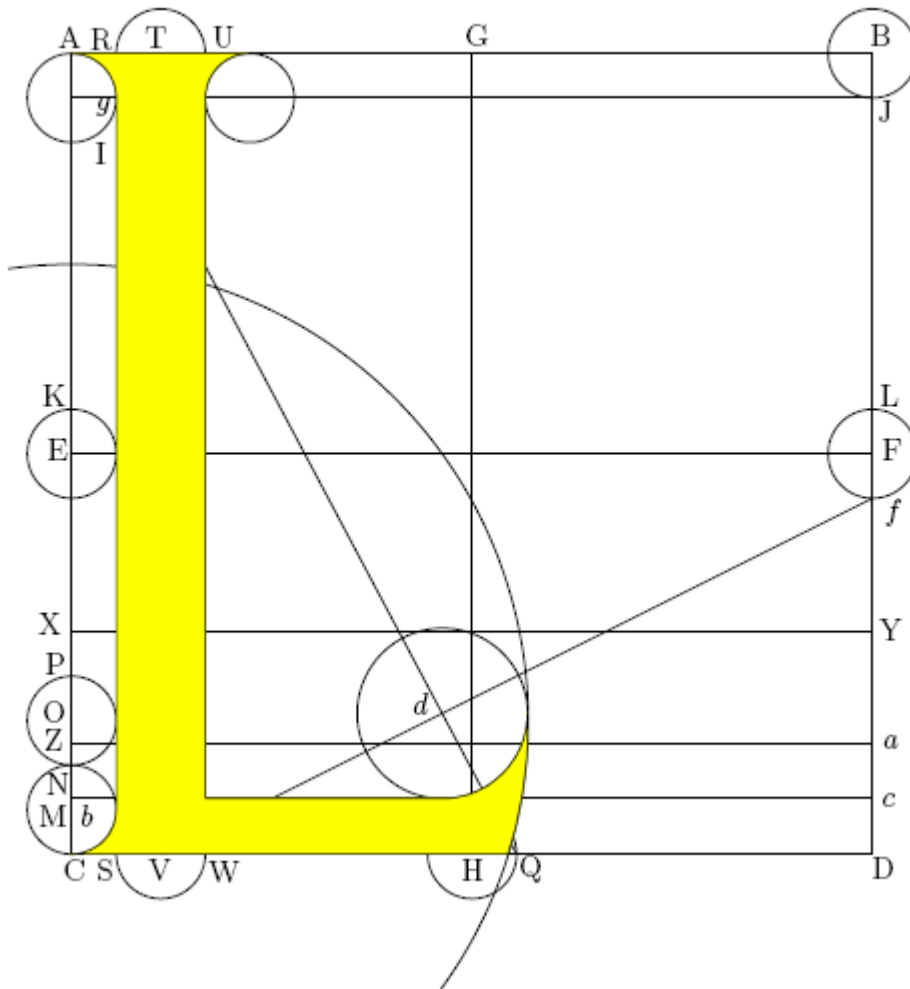
Simplification

- Implementation of [Lindstrom-Turk] volume-preserving method.



Summary: CGAL for Modeling and Polyhedral Surfaces

- The halfedge data structure and the polyhedron are highly flexible
- CGAL provides many algorithms for geometric modeling and geometry processing
- Polyhedral surface as output of surface mesh generation algorithms



CGAL for Mesh Generation

Pierre Alliez
INRIA

Outline

- 2D mesh generation
- Surface mesh generation
- 3D mesh generation
- Work in progress

2D Mesh Generation

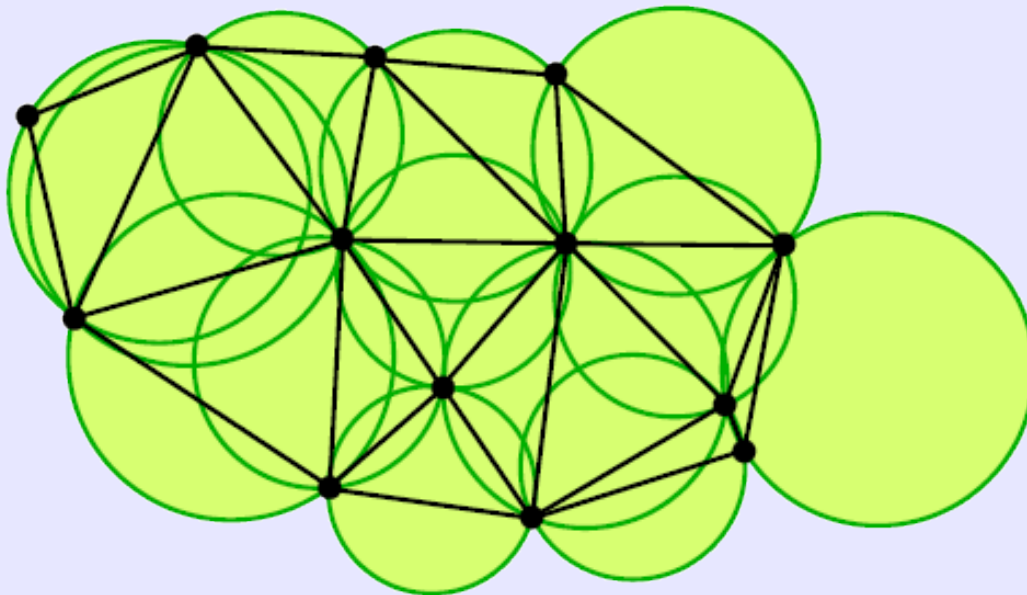
2D Mesh Generation



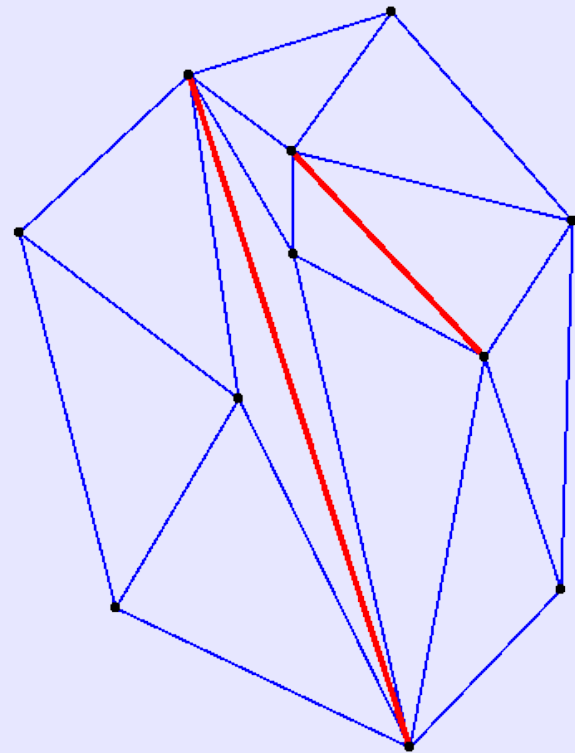
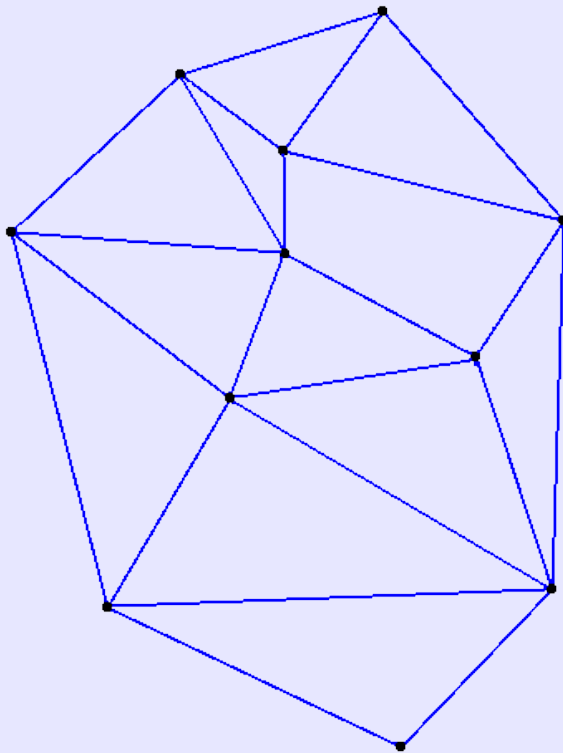
From Triangulations to Quality Meshes

Delaunay Triangulation

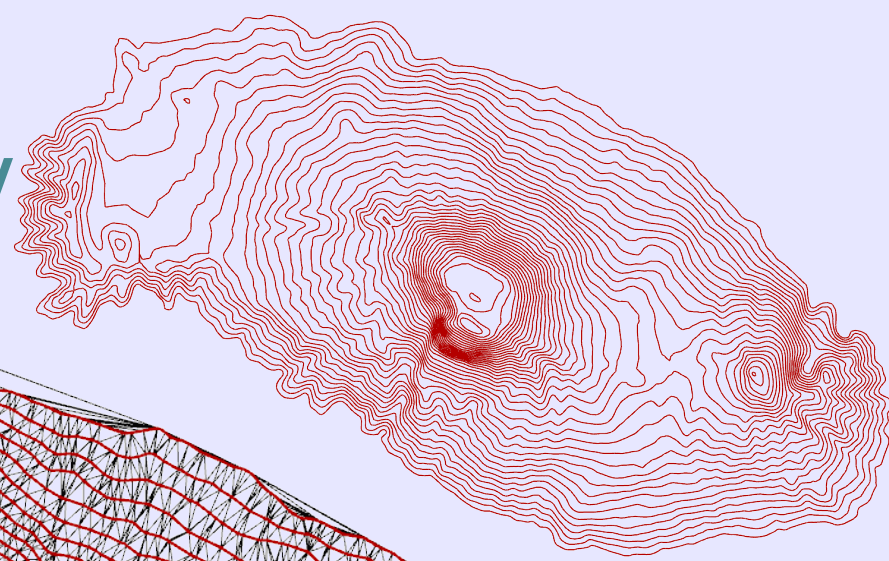
- A triangulation is a **Delaunay triangulation**, if the circumscribing circle of any facet of the triangulation contains no vertex in its interior



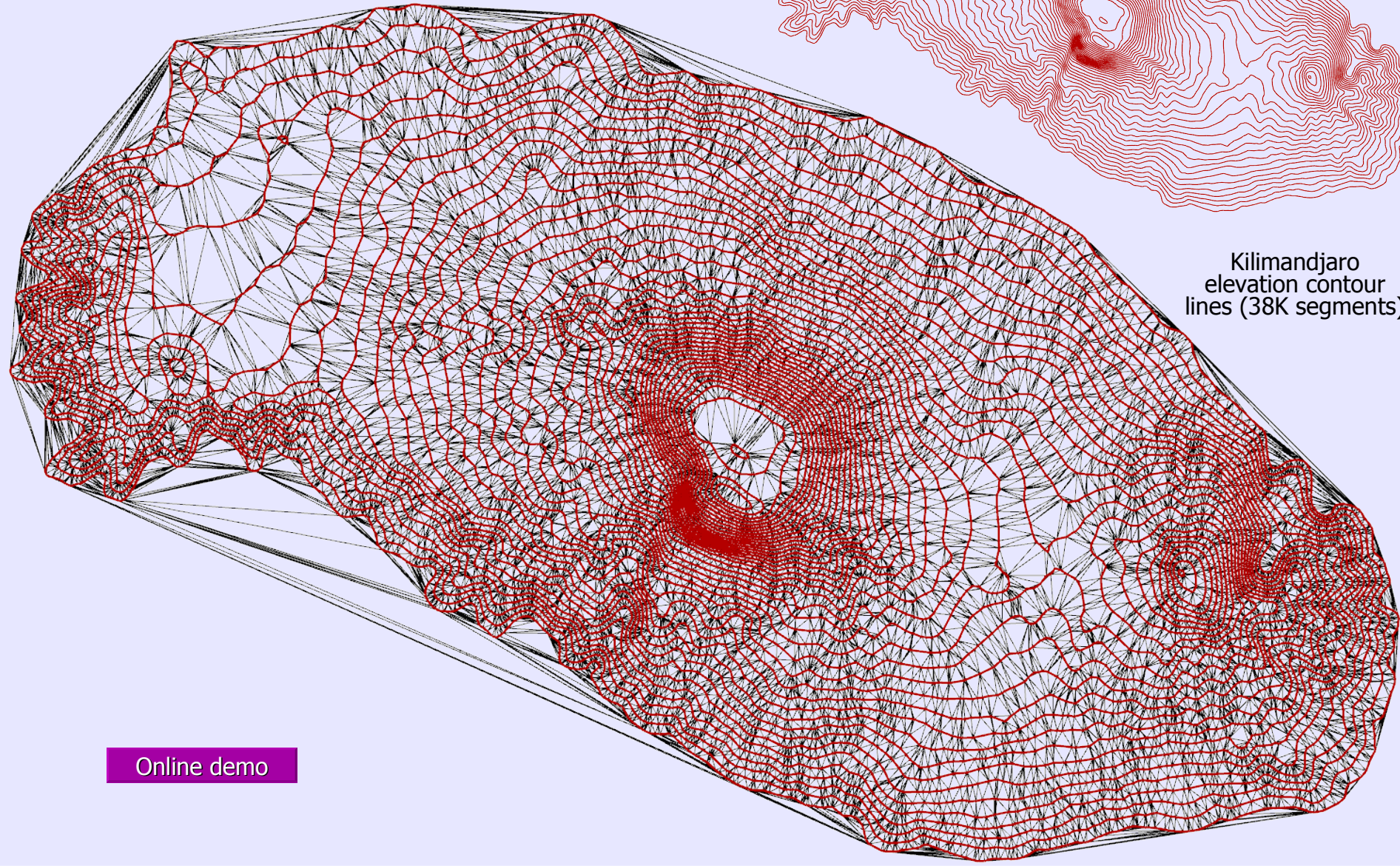
Constrained Delaunay Triangulation



Constrained Delaunay

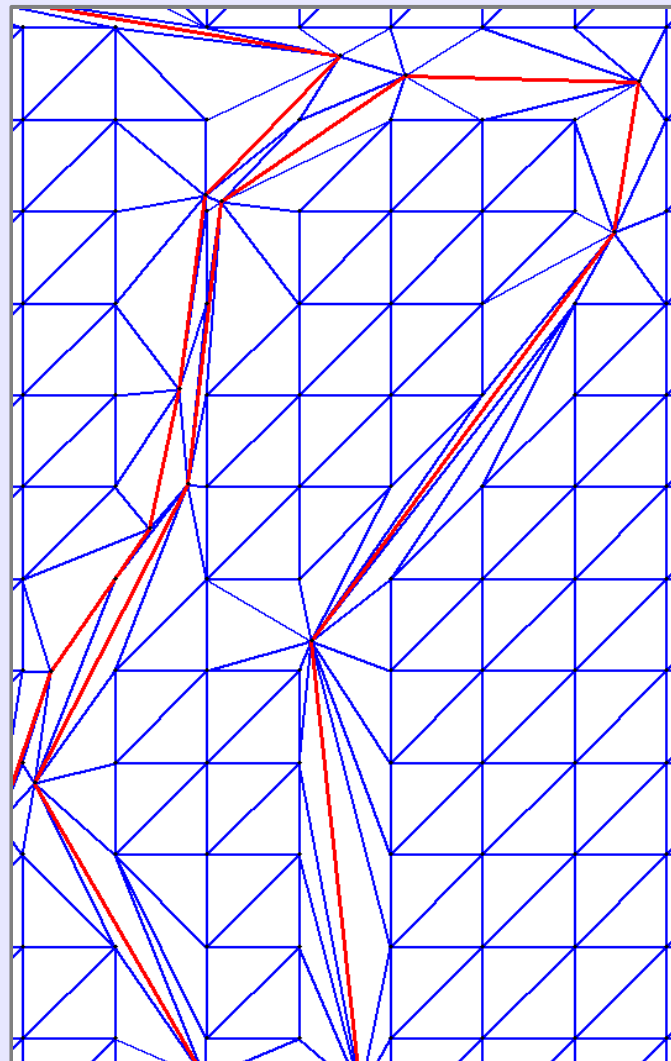
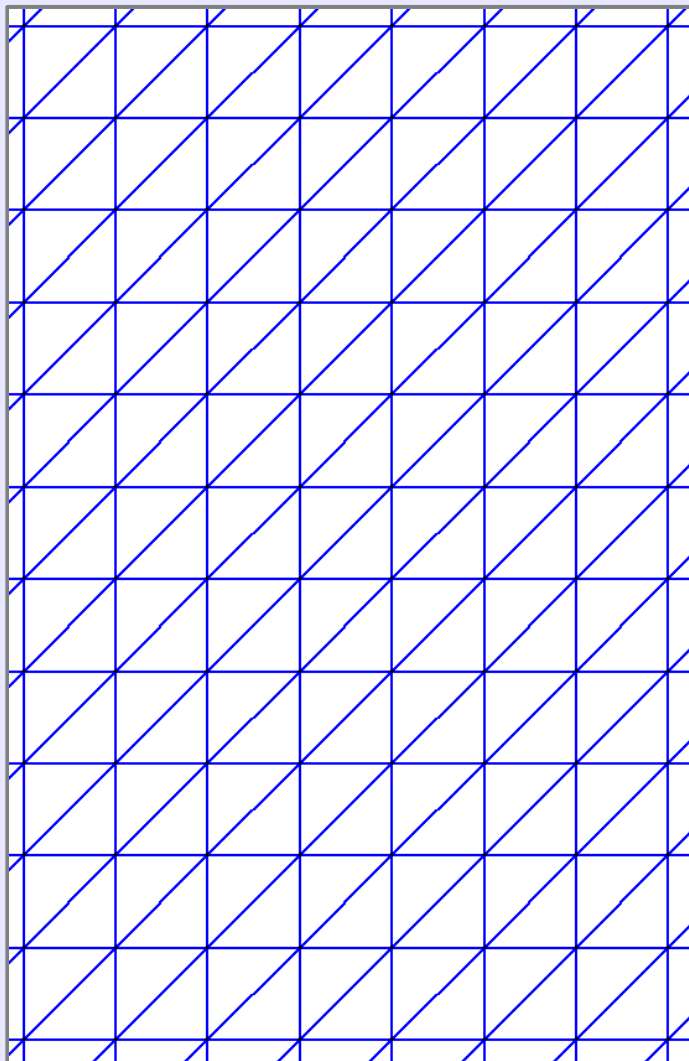


Kilimanjaro
elevation contour
lines (38K segments)

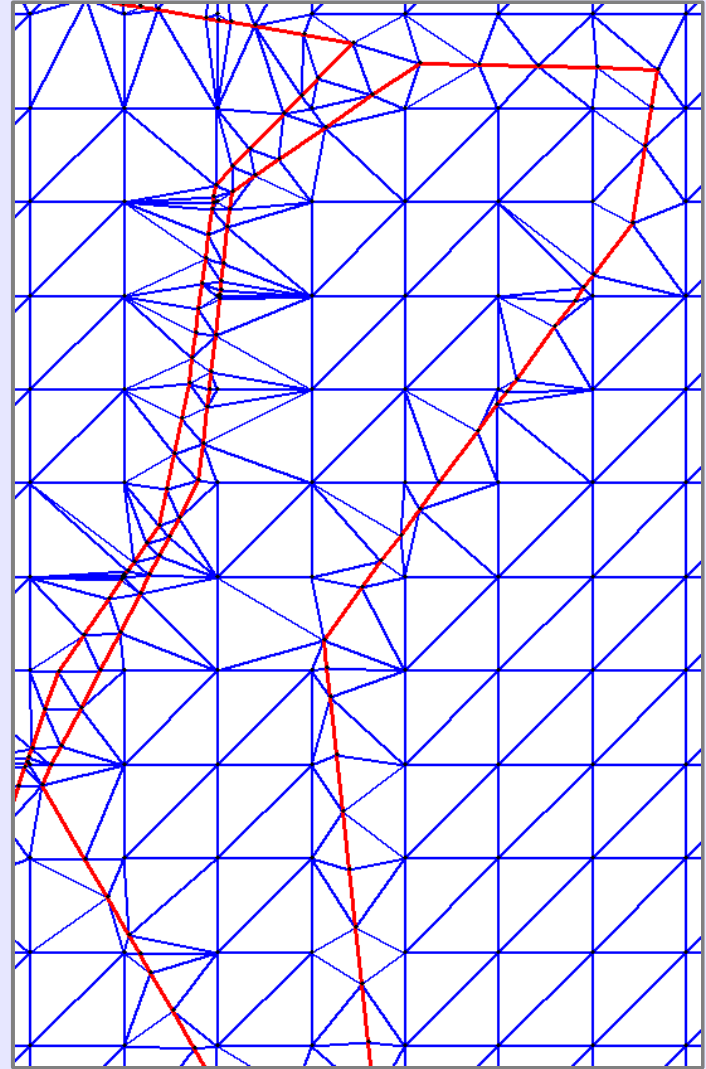
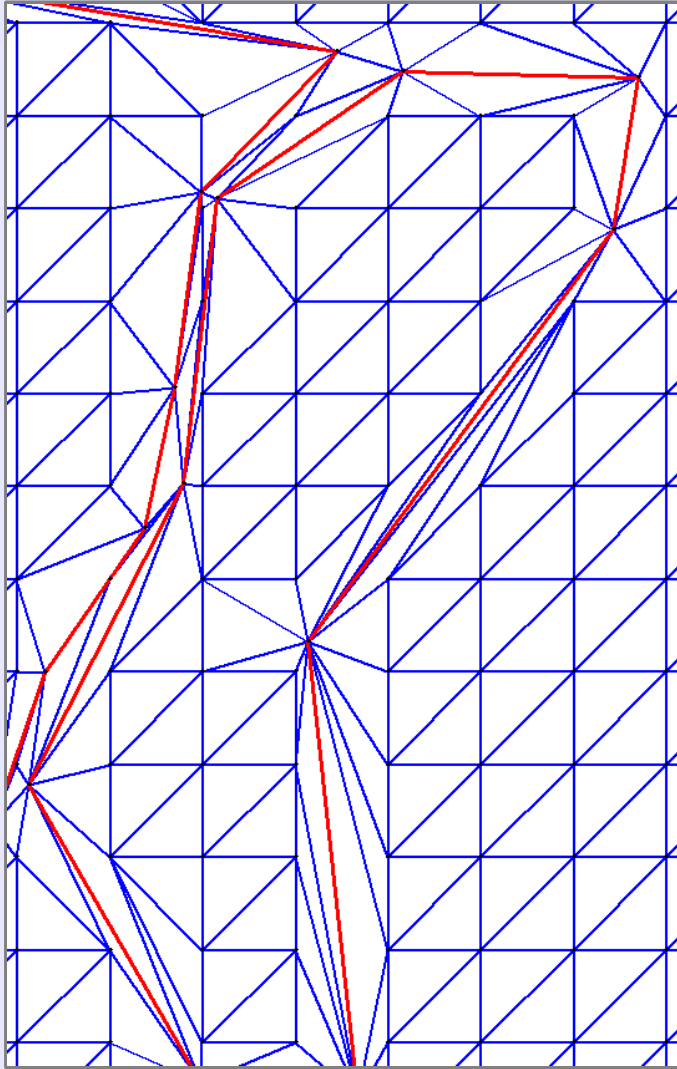


[Online demo](#)

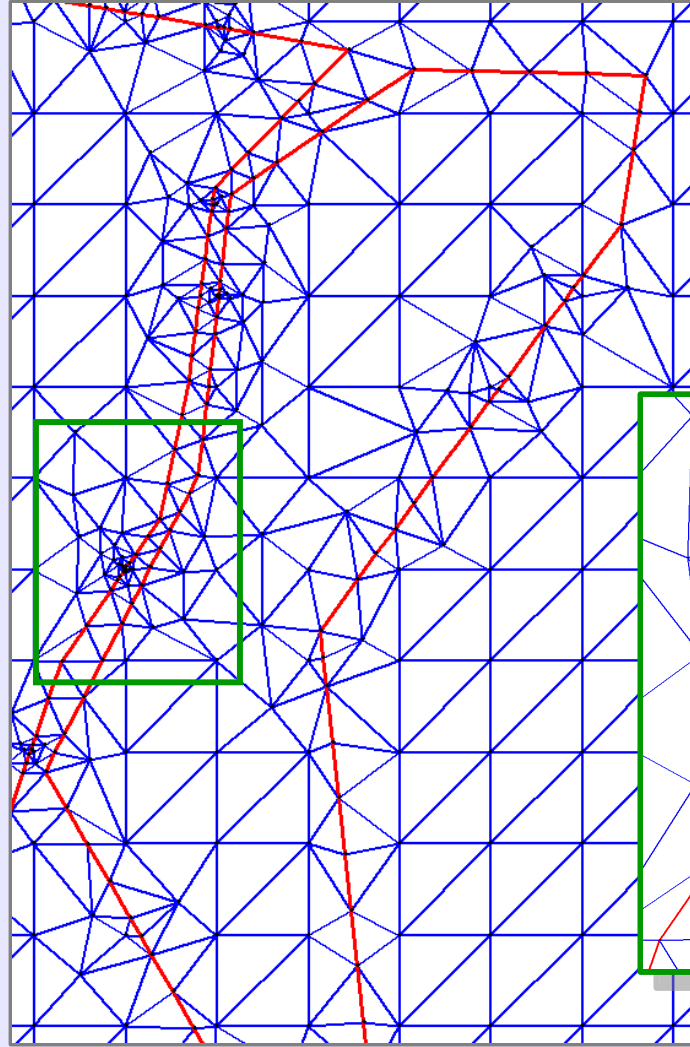
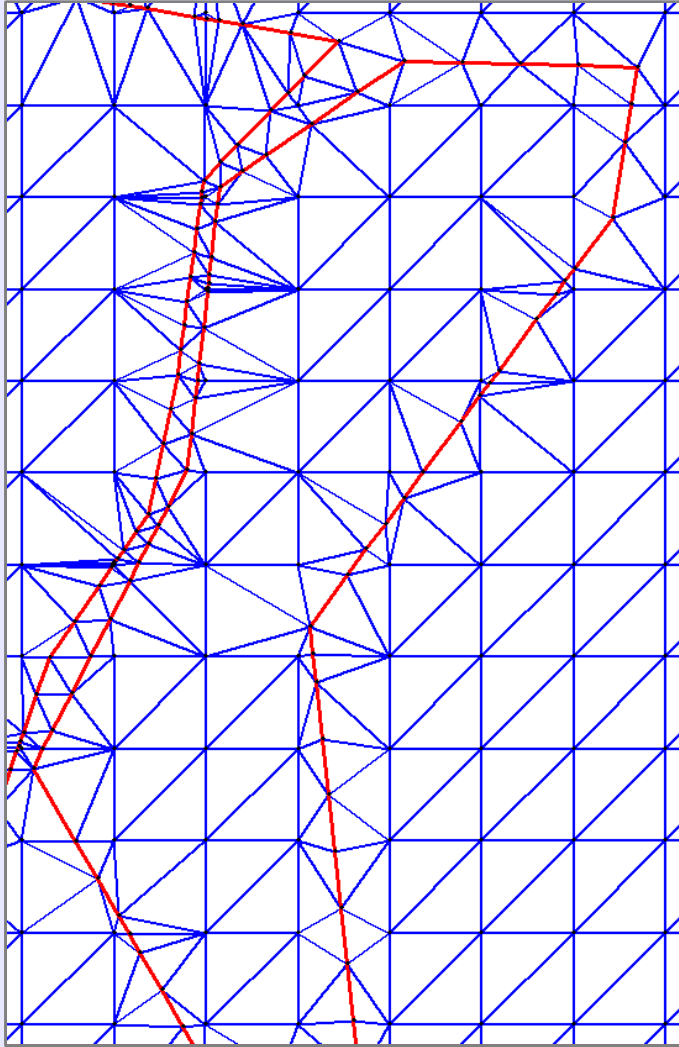
Adding Constraints



Conforming a Triangulation

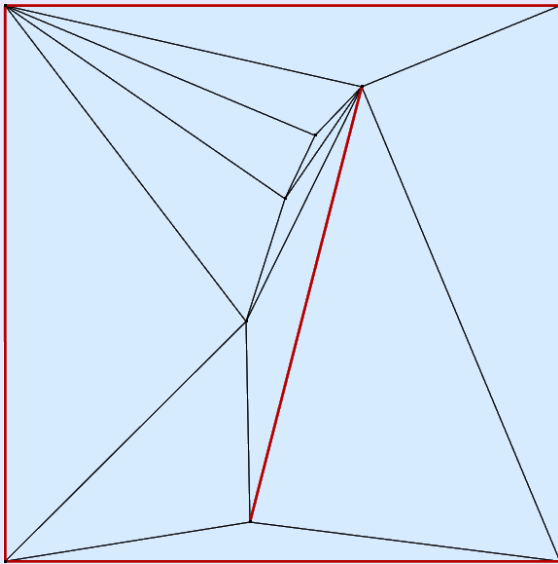


Delaunay Refinement

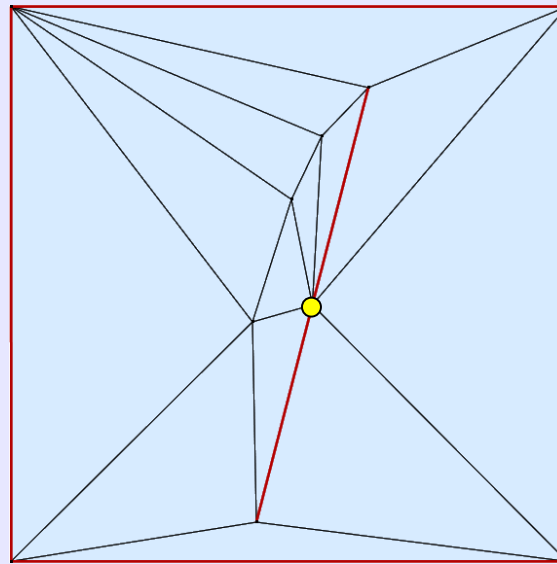


Conforming a Triangulation

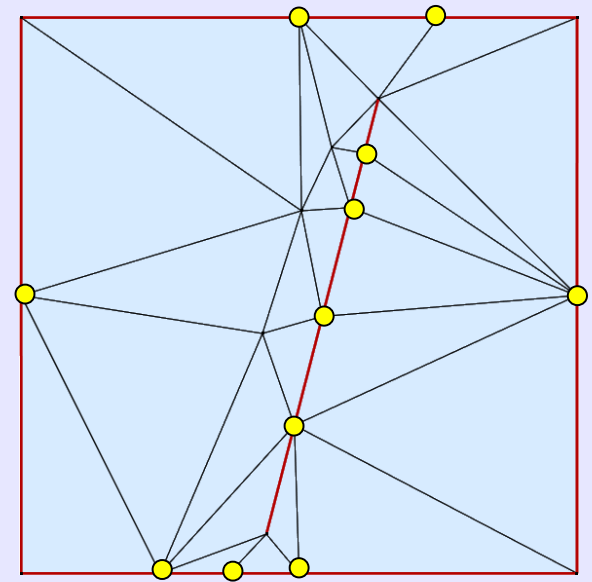
Any constrained Delaunay triangulation can be **refined** into a conforming Delaunay or Gabriel triangulation by adding **Steiner vertices**.



non conforming



Delaunay conforming

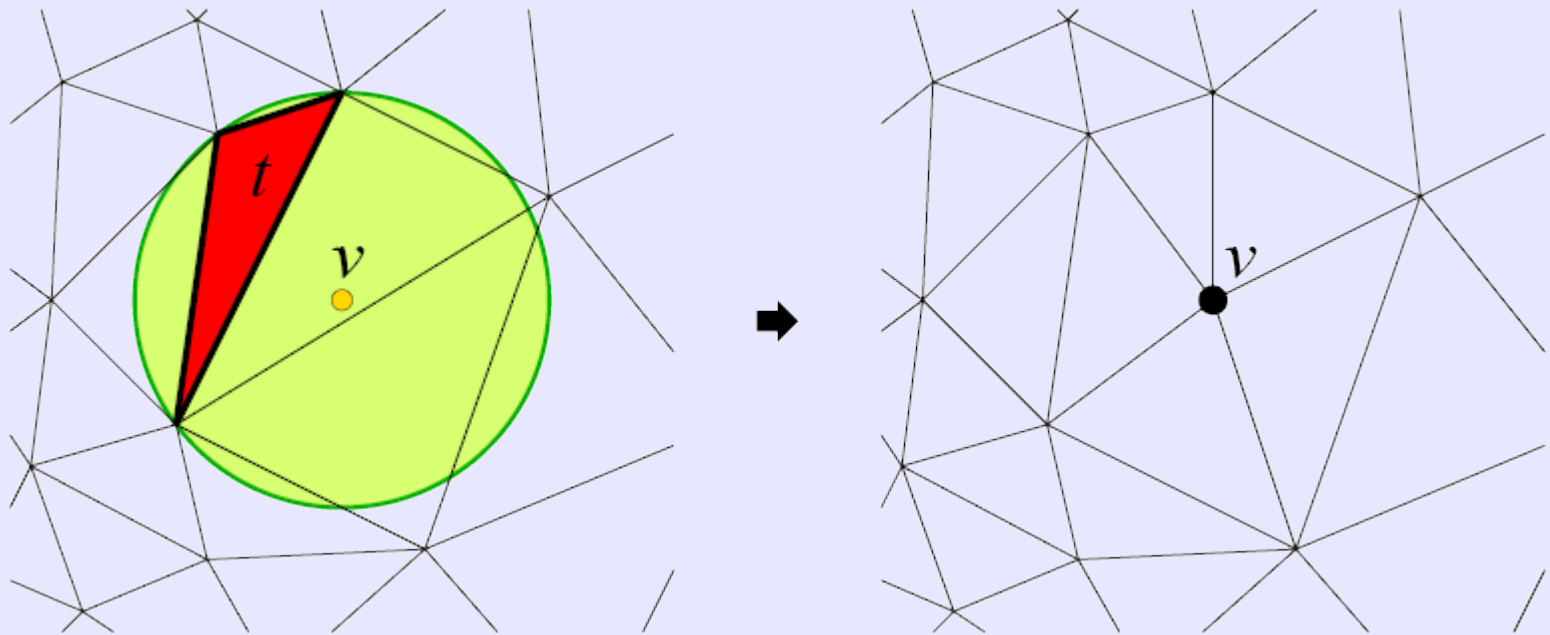


Gabriel conforming

Delaunay Refinement Rules

Rule #1: break **bad** elements by inserting circumcenters (Voronoi vertices)

- “bad” in terms of **size** or **shape** (too big or skinny)

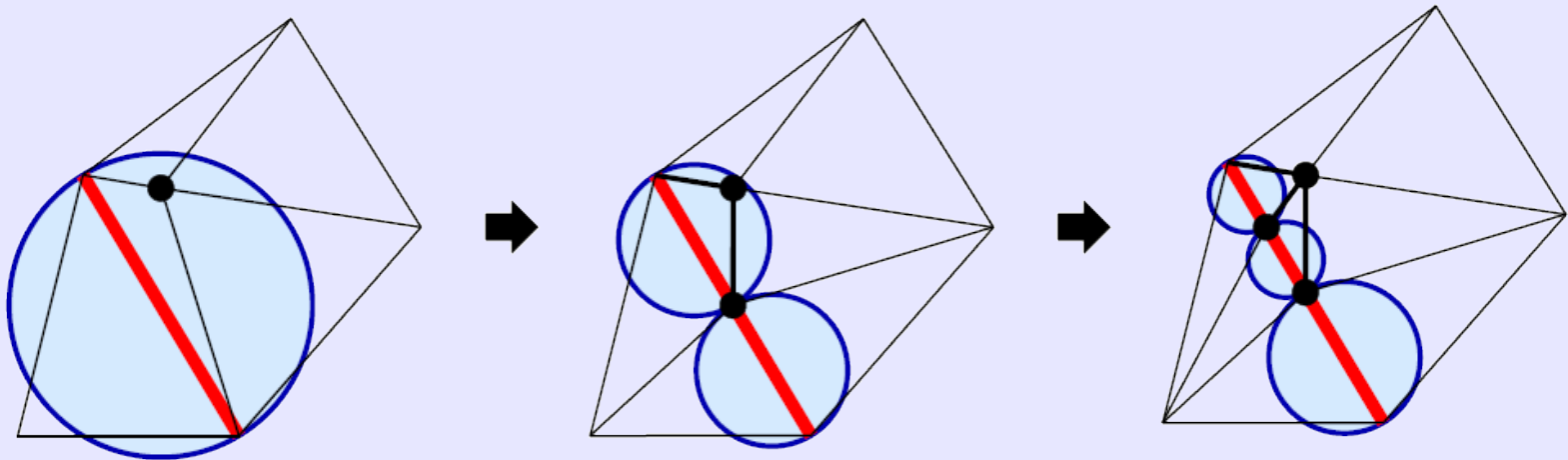


Picture taken from [Shewchuk]

Delaunay Refinement Rules

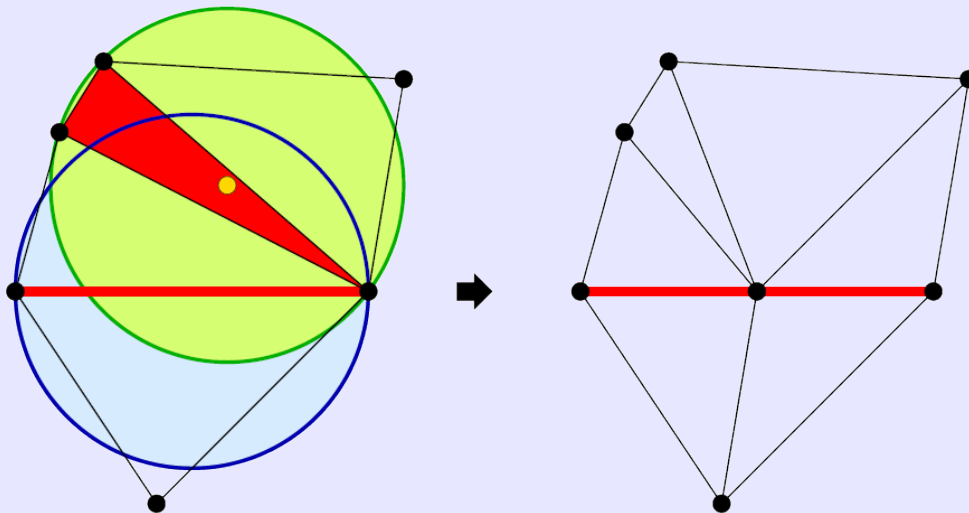
Rule #2: Midpoint vertex insertion

A constrained segment is said to be **encroached**, if there is a vertex inside its diametral circle



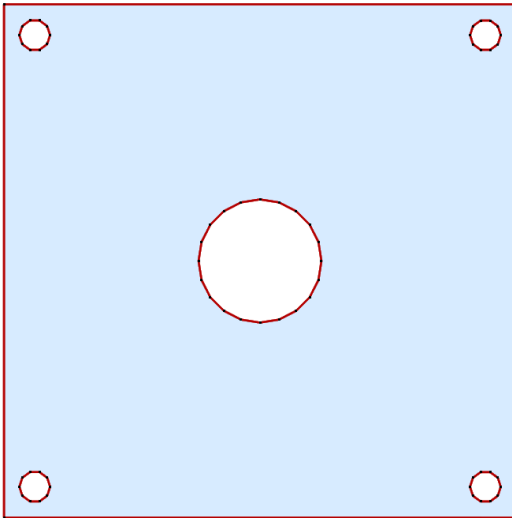
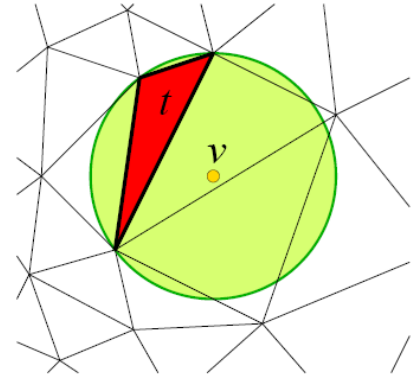
Delaunay Refinement Rules

Encroached subsegments have priority over skinny triangles

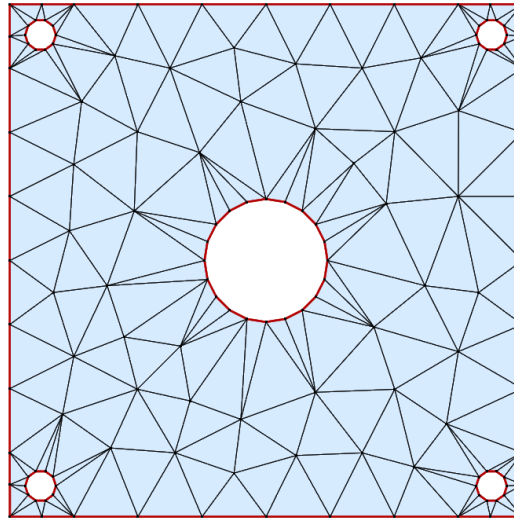


Parameters for Delaunay Refinement

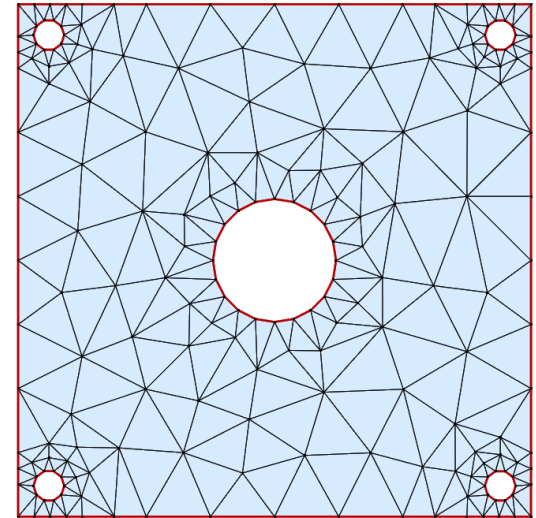
- **Shape**
 - Lower bound on triangle angles



Input PLSG



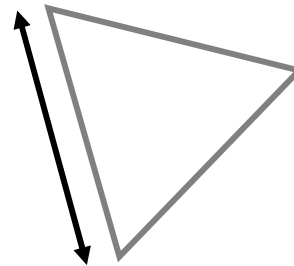
5 deg



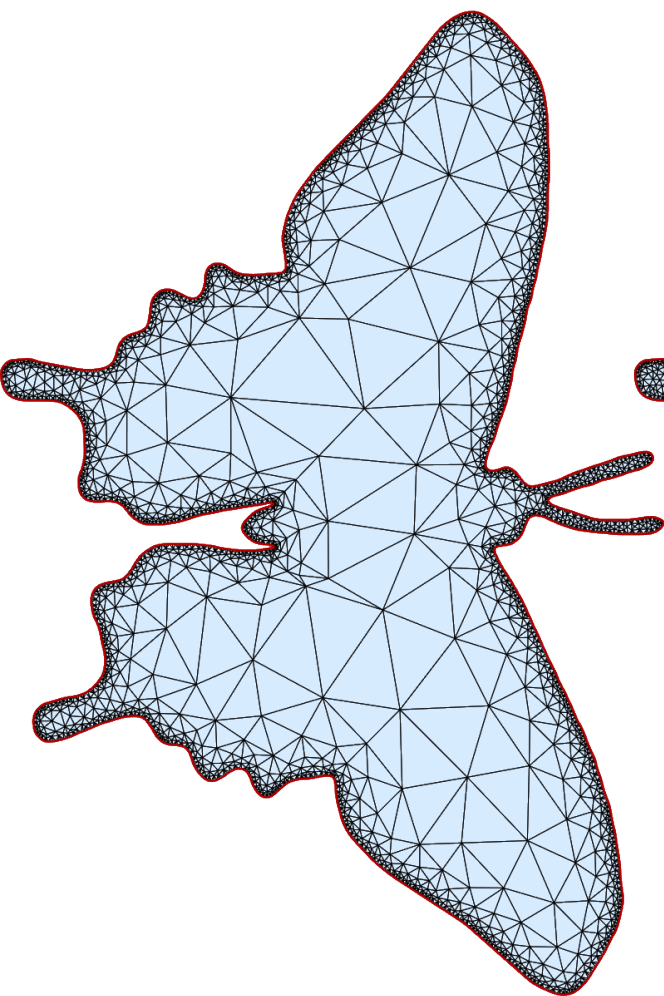
20.7 deg

Parameters for Delaunay Refinement

- Shape
 - Lower bound on triangle angles
- **Size**
 - No constraint
 - Uniform sizing
 - Sizing function



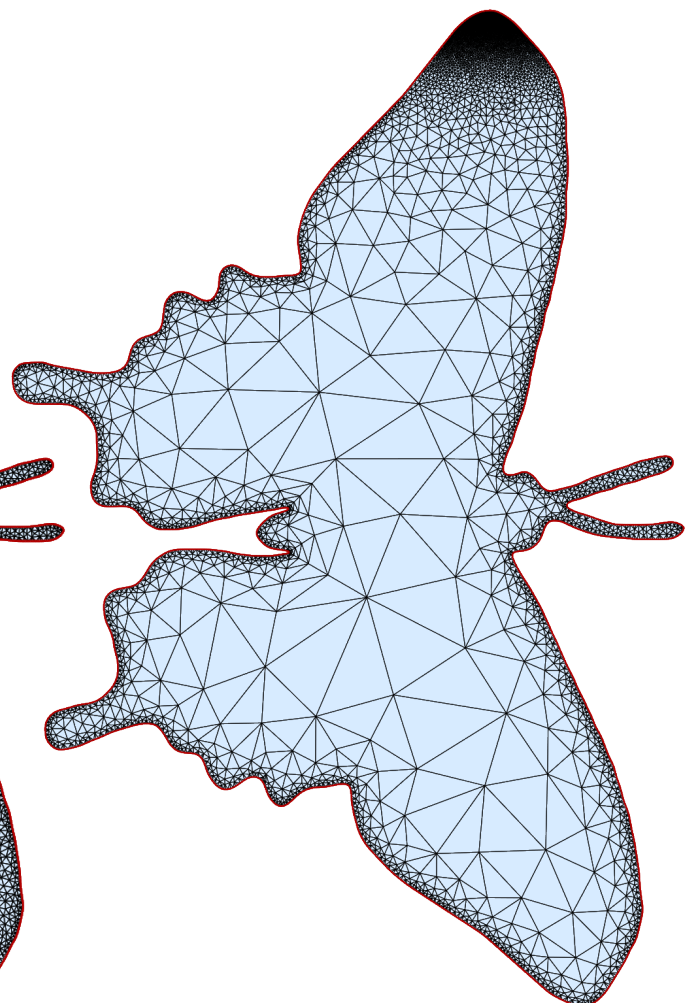
Sizing Parameter



No constraint



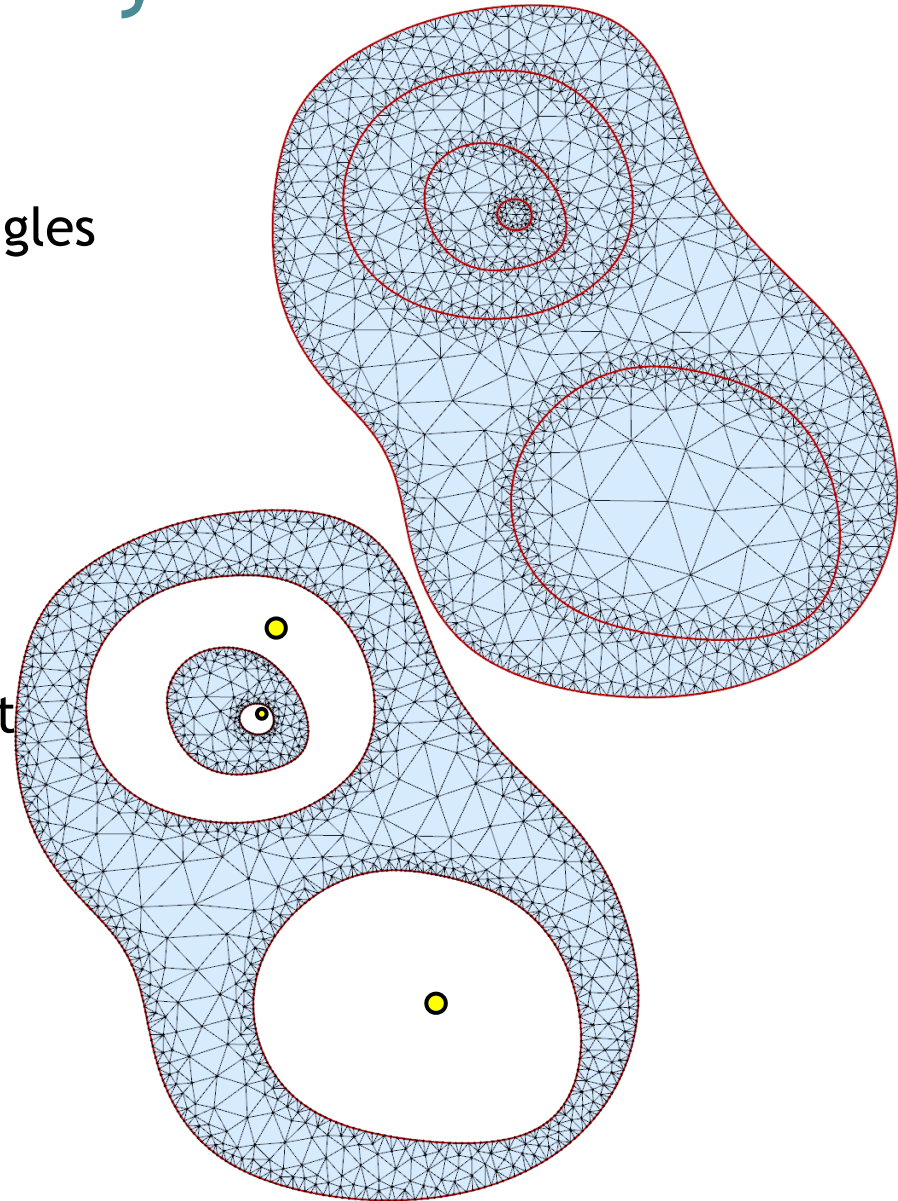
Uniform



Sizing function

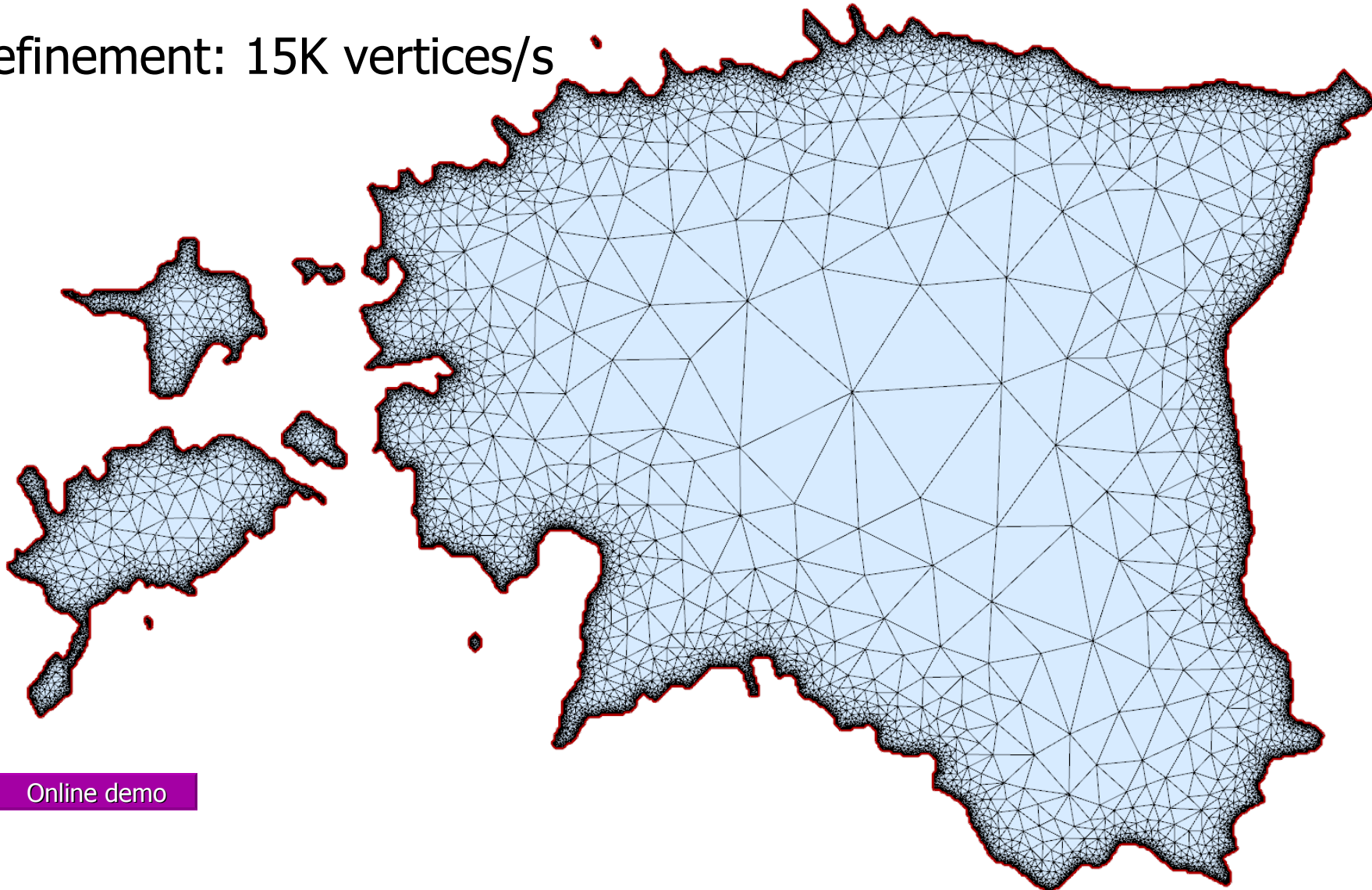
Parameters for Delaunay Refinement

- **Shape**
 - Lower bound on triangle angles
- **Size**
 - No constraint
 - Uniform sizing
 - Sizing function
- **Seeds**
 - Exclude/include component



Performances

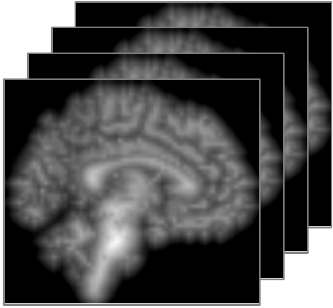
Refinement: 15K vertices/s



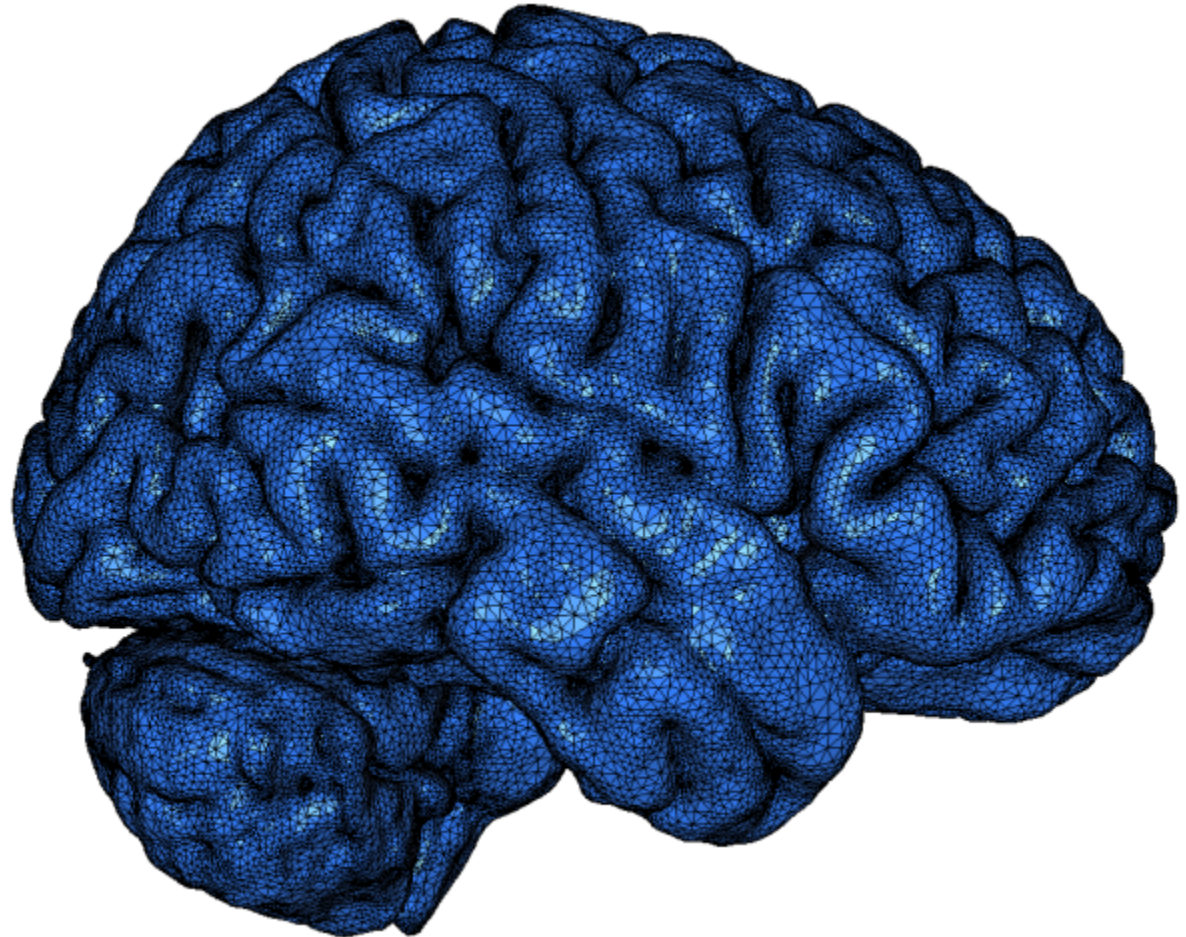
Online demo

Surface Mesh Generation

Surface Mesh Generation

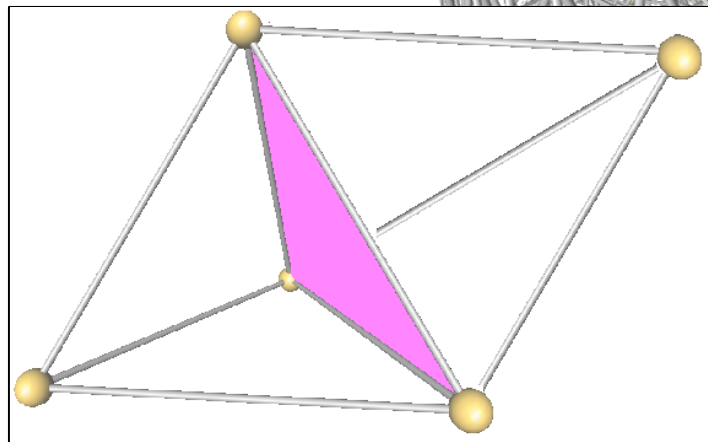


input



3D Triangulations

- Delaunay
- Regular
- Rich API
- Fully dynamic
- 1M points in 16s

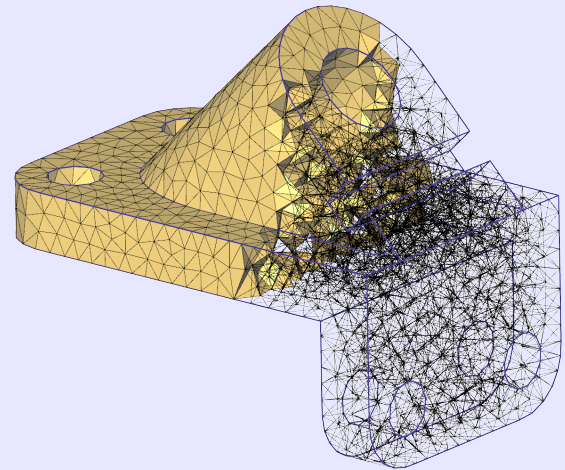
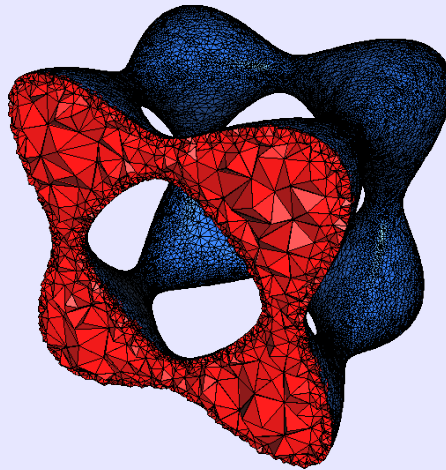
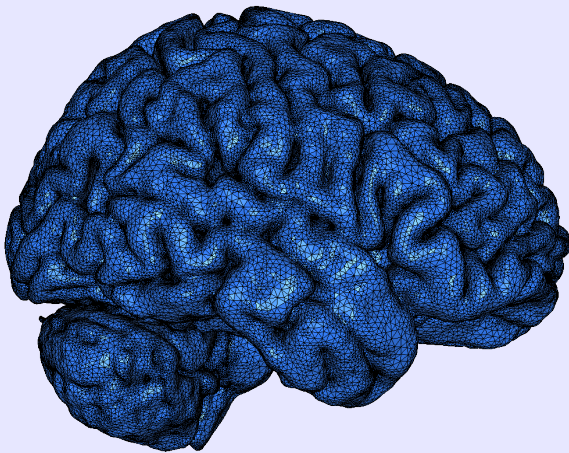


[Online manual](#)

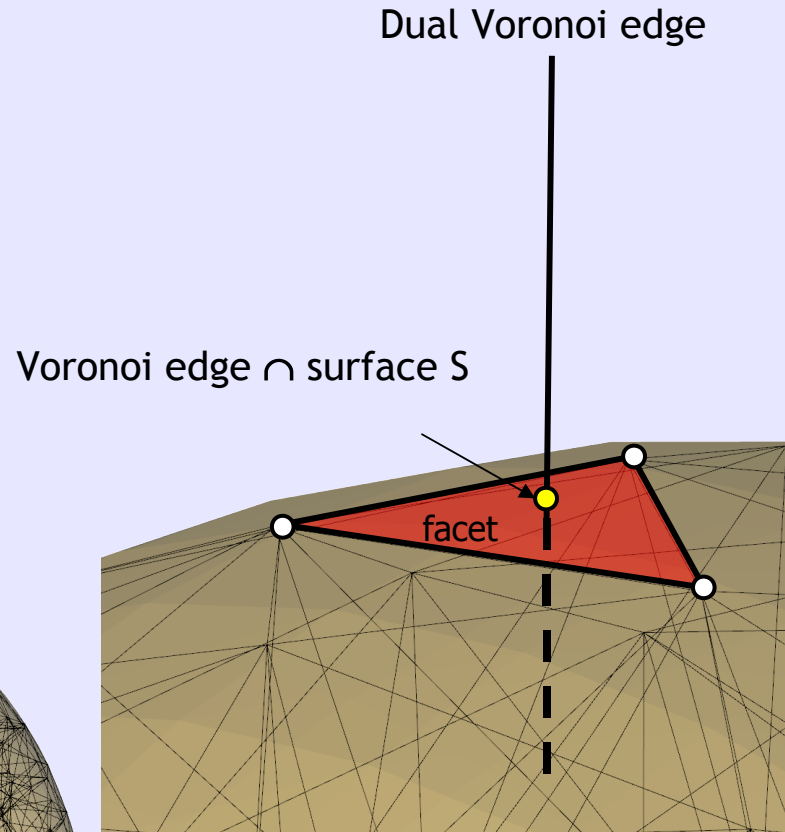
Mesh Generation

Key concepts:

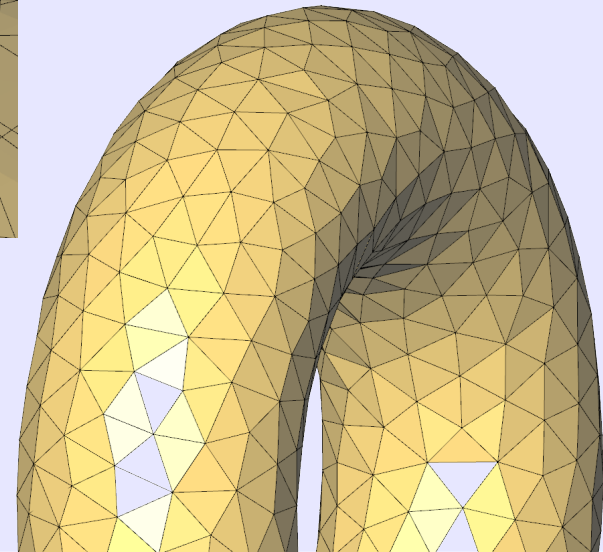
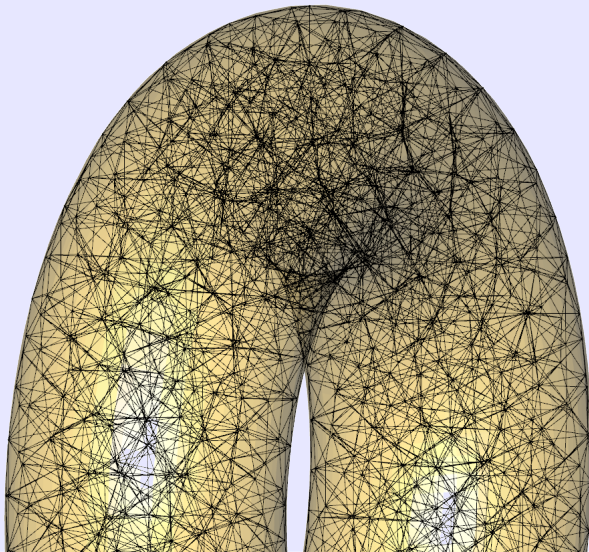
- Delaunay **filtering**
- Delaunay **refinement**



Delaunay Filtering



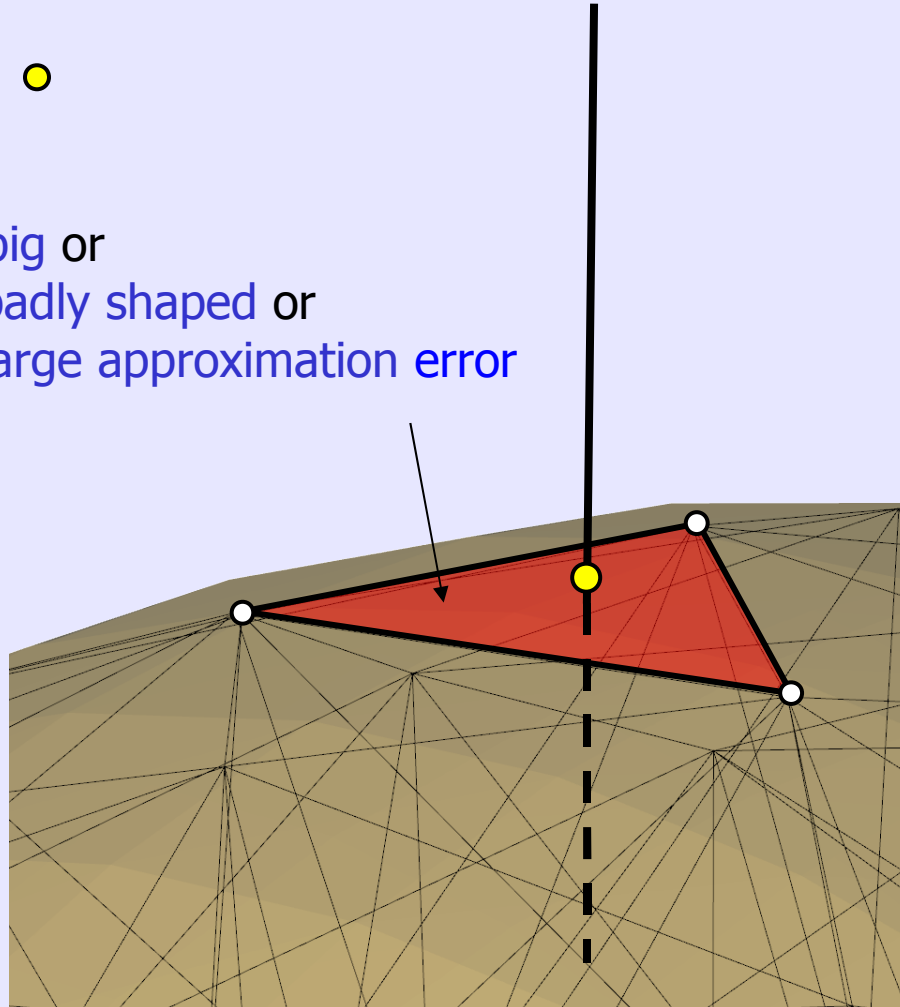
**Delaunay
triangulation
restricted to
surface S**



Delaunay Refinement

Steiner point ●

Bad facet = big or
badly shaped or
large approximation error



Delaunay Refinement

repeat

{

pick bad facet f

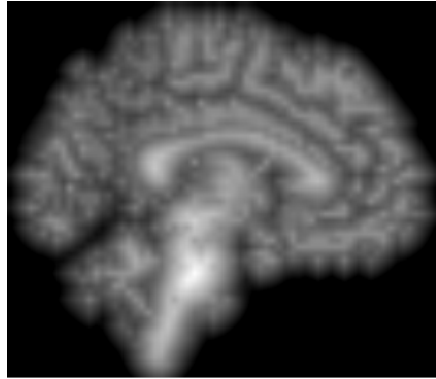
insert furthest $(\text{dual}(f) \cap S)$ in Delaunay triangulation

update Delaunay triangulation restricted to S

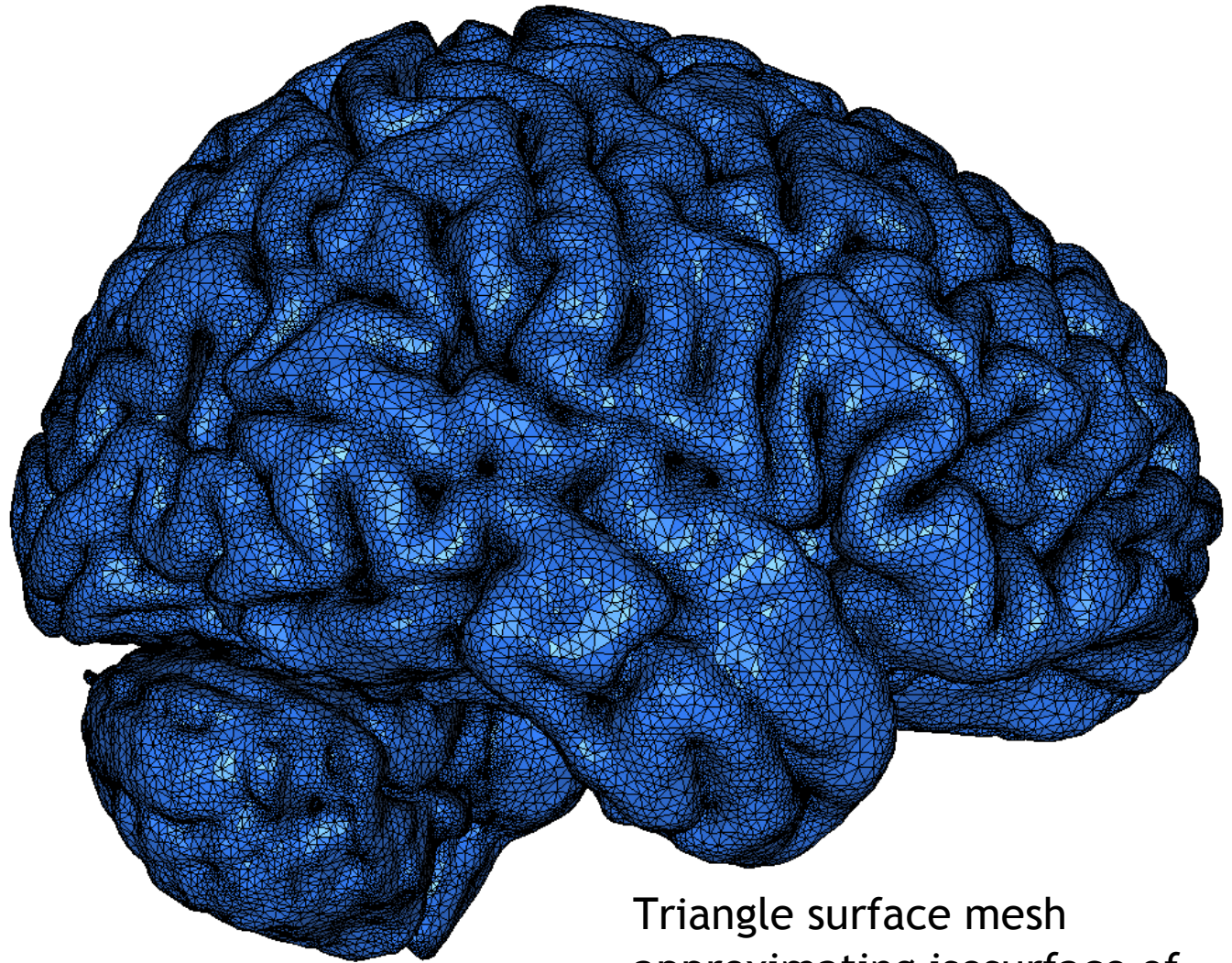
}

until all facets are good

Output Mesh



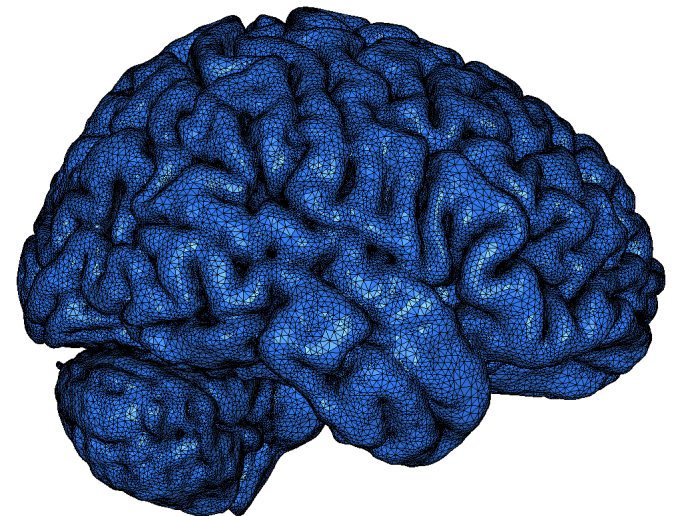
input



Triangle surface mesh
approximating isosurface of
input 3D image

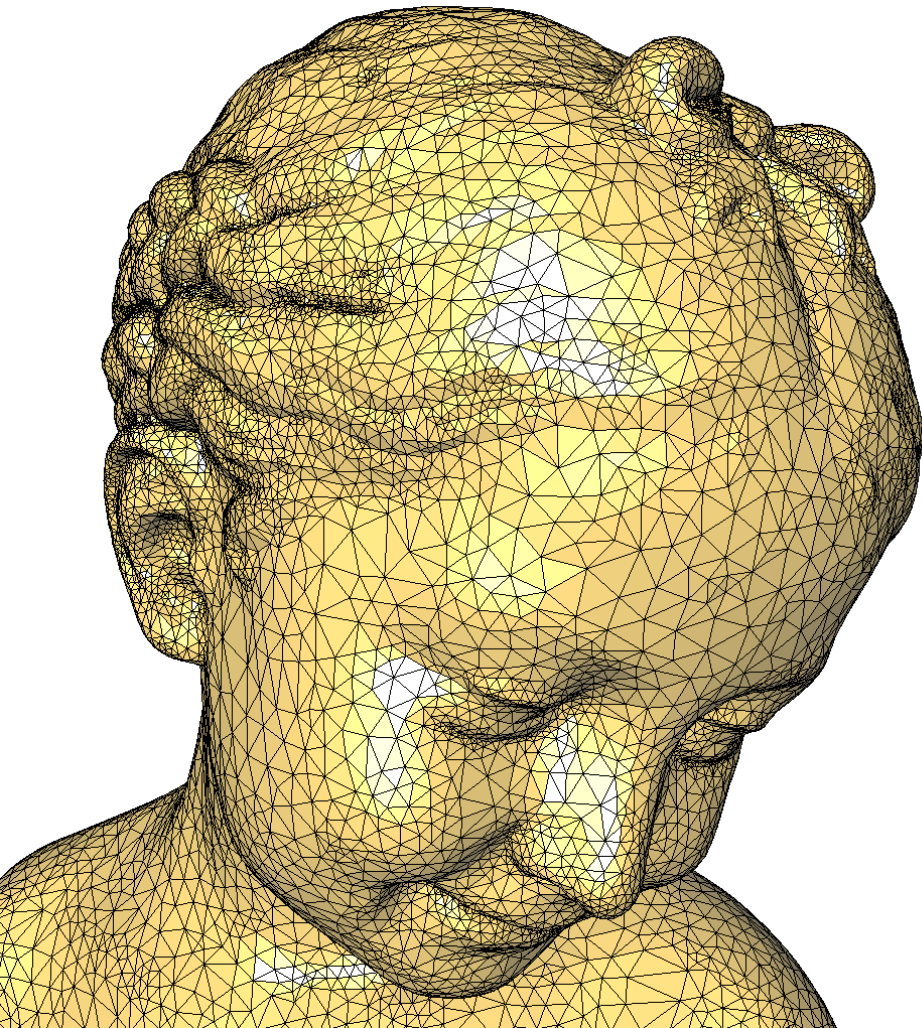
Output Mesh

- Well shaped triangles
 - Lower bound on triangle angles
- Homeomorphic to input surface
- Manifold
 - not only combinatorially, i.e., no self-intersection
- Faithful Approximation of input surface
 - Hausdorff distance
 - Normals

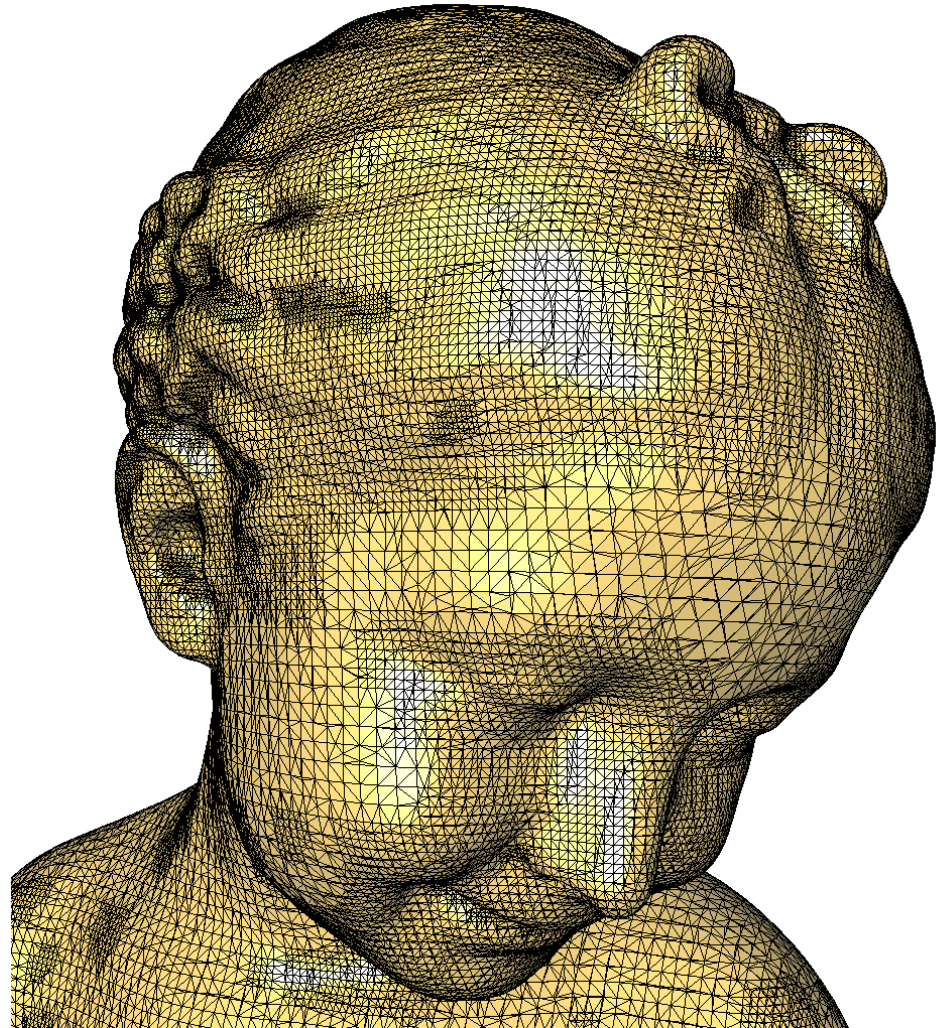


vs Marching Cubes

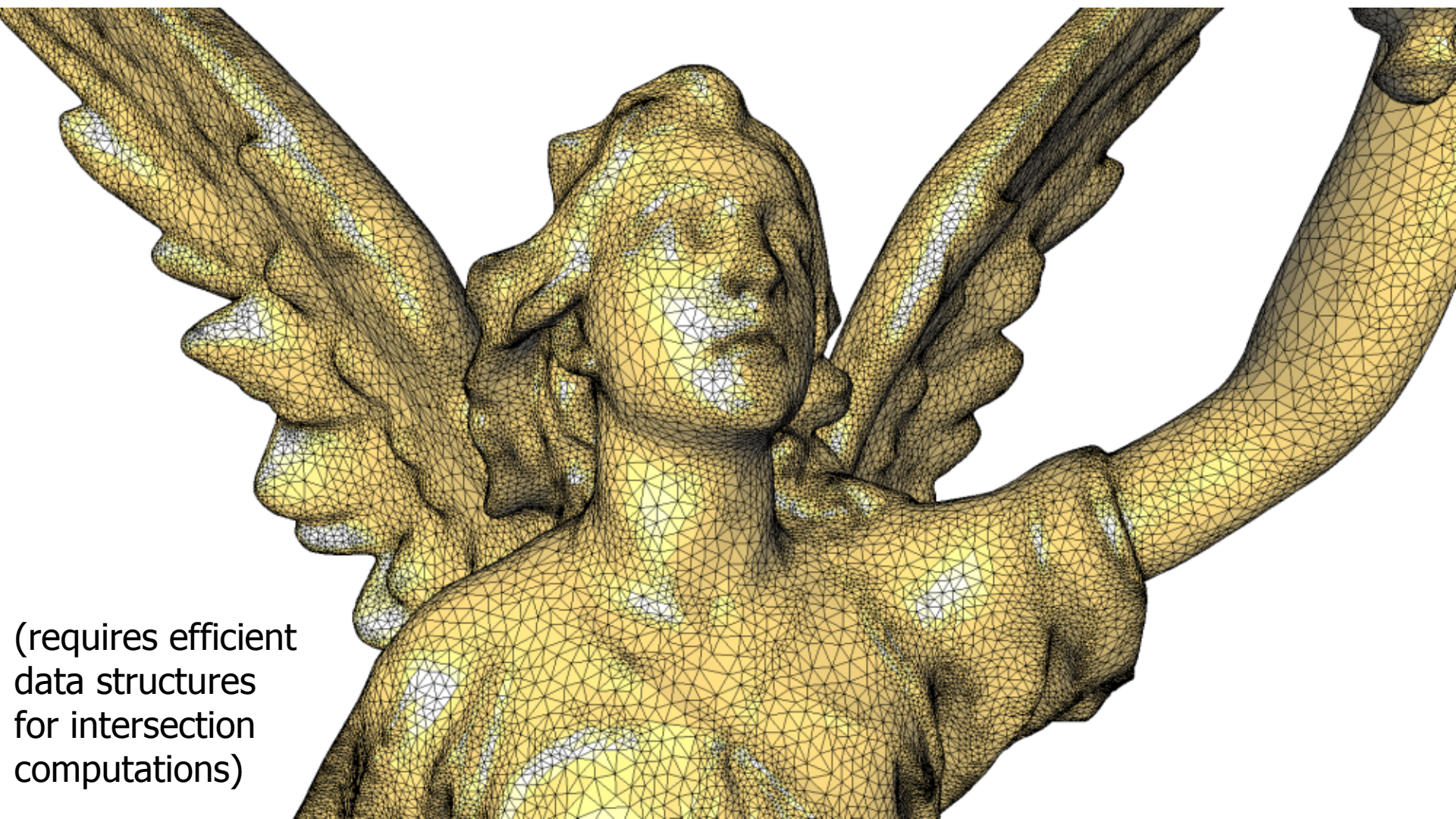
Delaunay refinement



Marching cubes in octree



Surface Remeshing (input is a polyhedral surface)



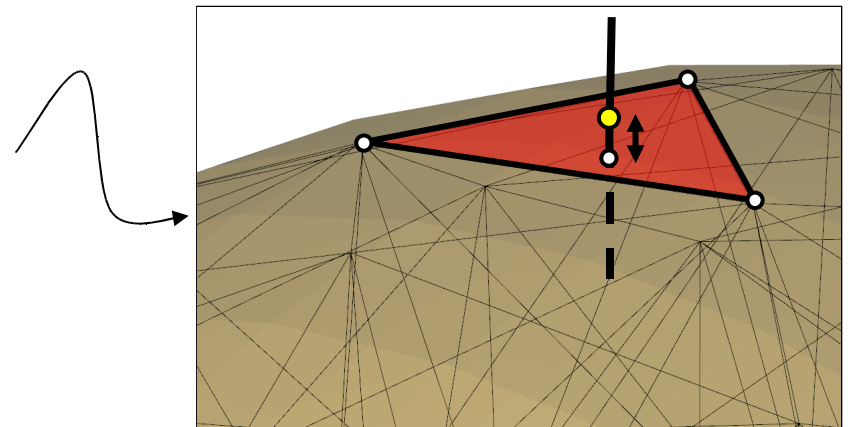
(requires efficient
data structures
for intersection
computations)

Parameters

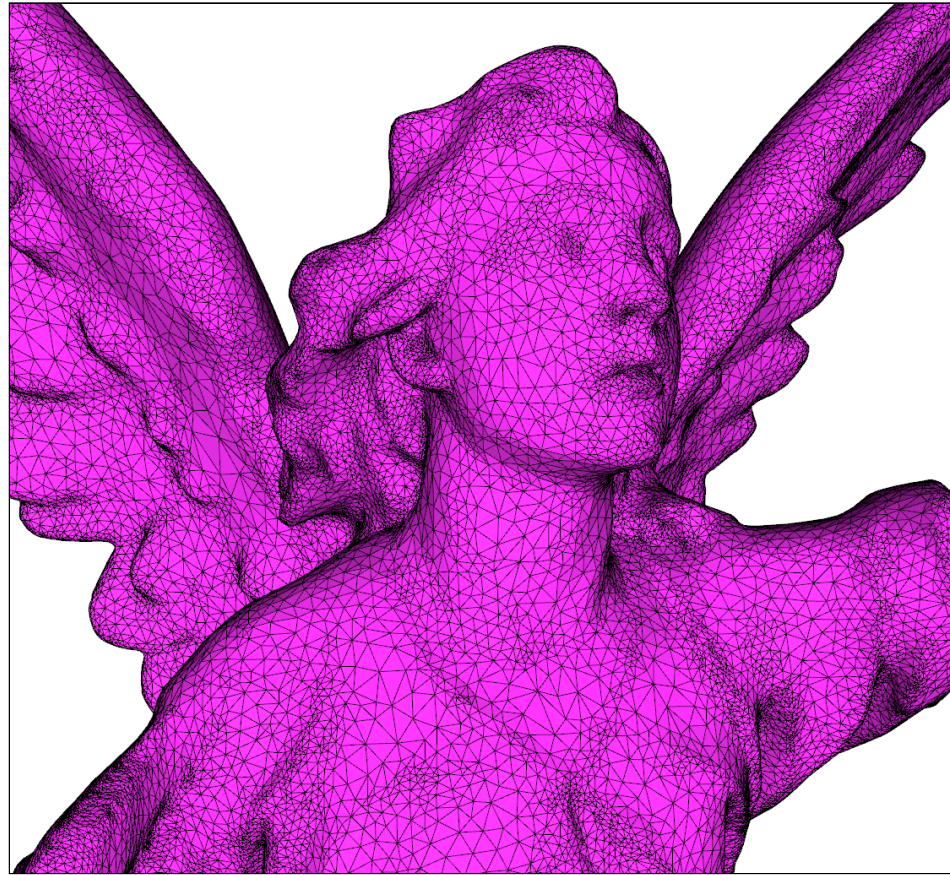
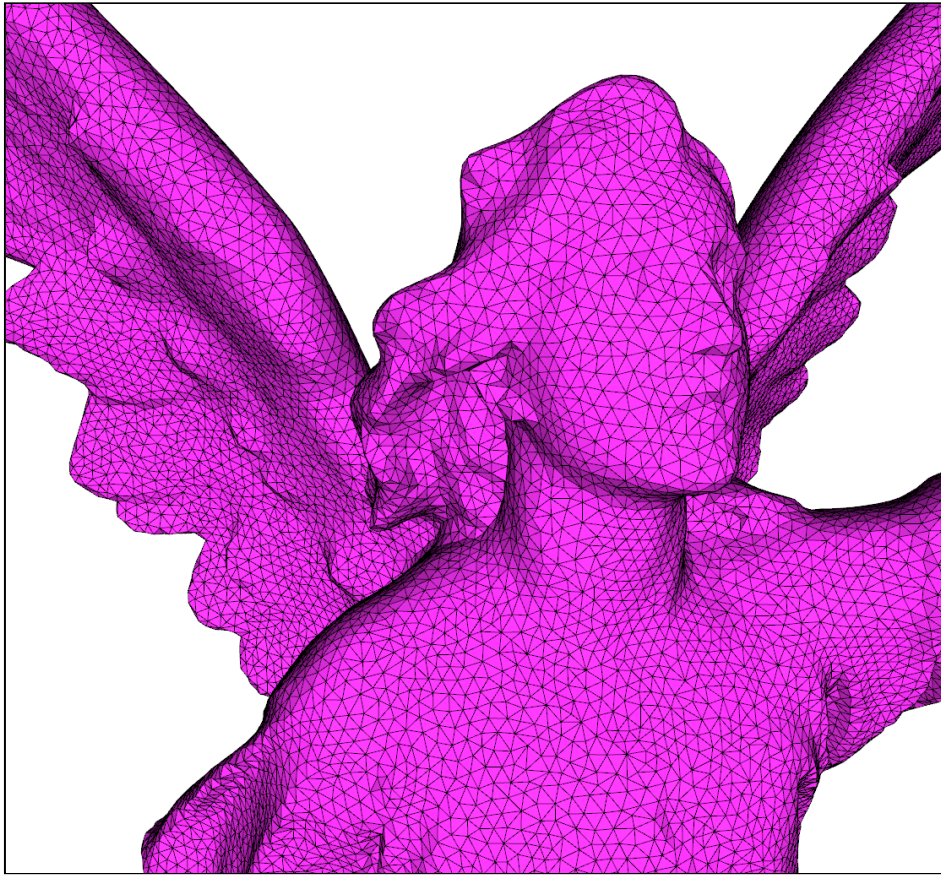
- Shape of triangles
 - lower bound on triangle angles
- Size
 - No constraint
 - Uniform sizing
 - Sizing function

Parameters

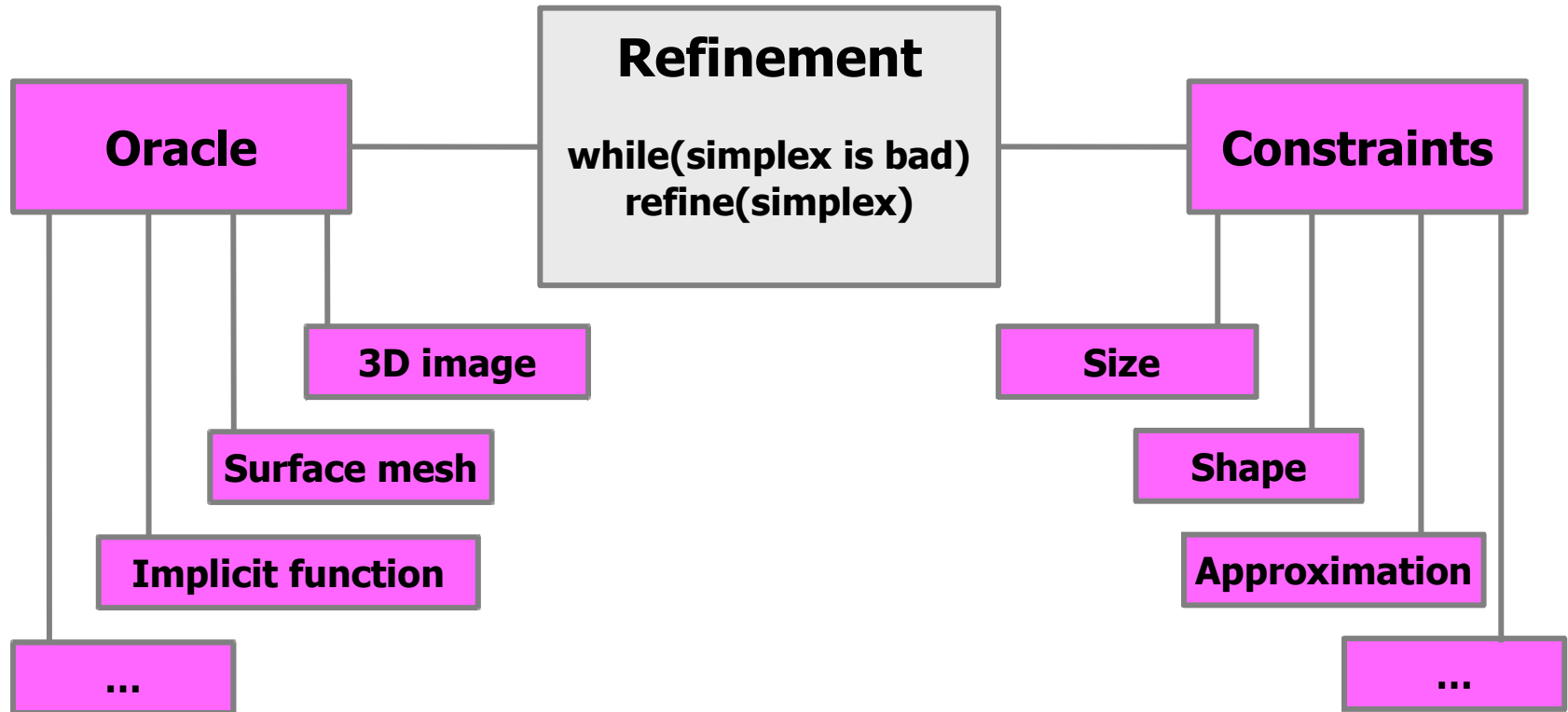
- Shape of triangles
 - lower bound on triangle angles
- Size
 - No constraint
 - Uniform sizing
 - Sizing function
- Approximation error



Uniform vs Adapted



Mesh Generation Framework

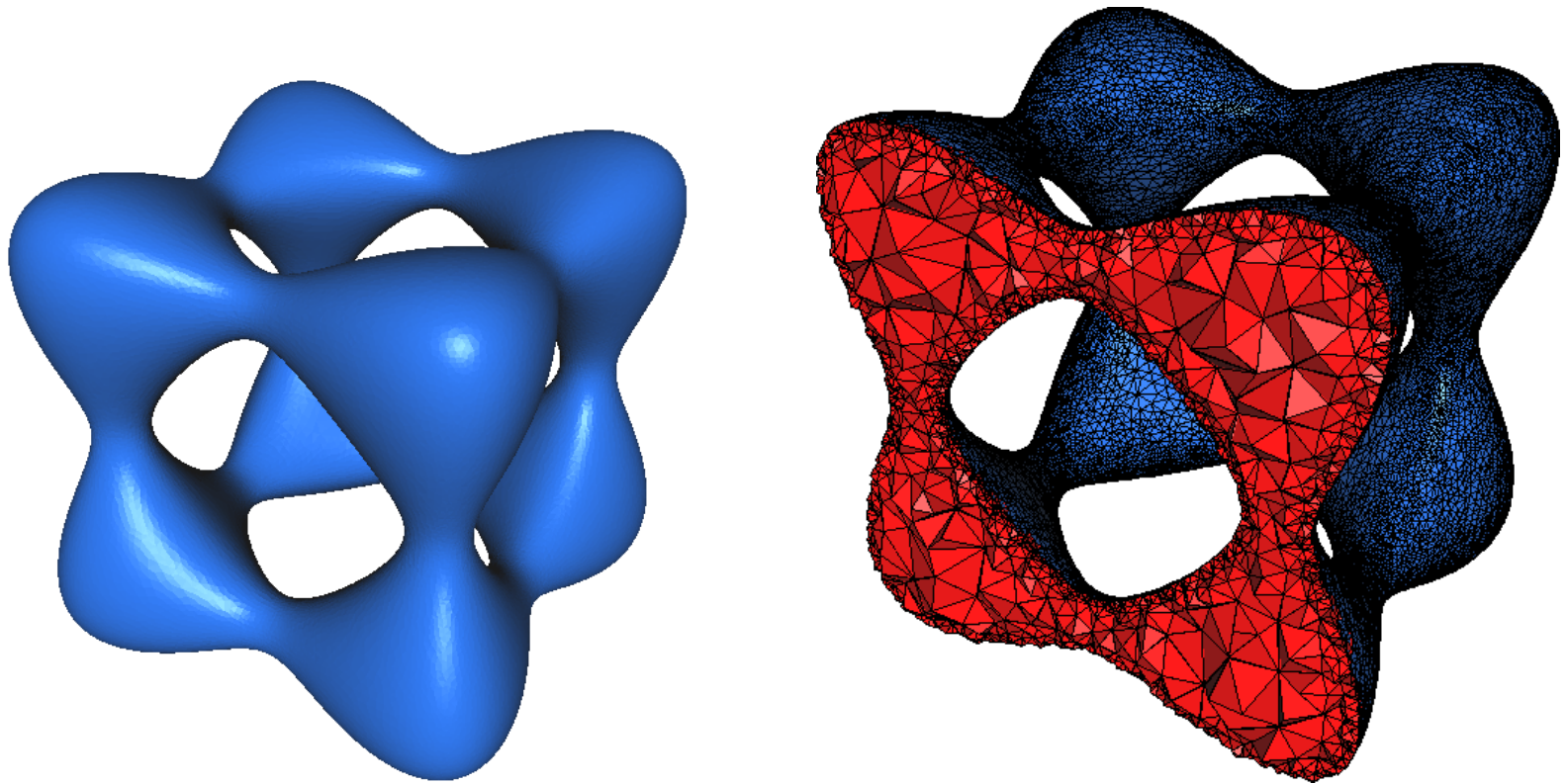


A Versatile Framework

- 3D grey level images
- 3D multi-domain images
- Implicit function: $f(x, y, z) = \text{constant}$
- Surface mesh (remeshing)
- Point set (surface reconstruction)
- Anything which provides intersections

3D Mesh Generation

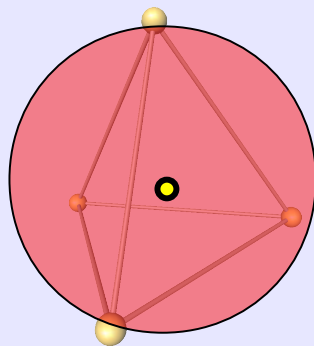
3D (Volume) Mesh Generation



More Delaunay Filtering

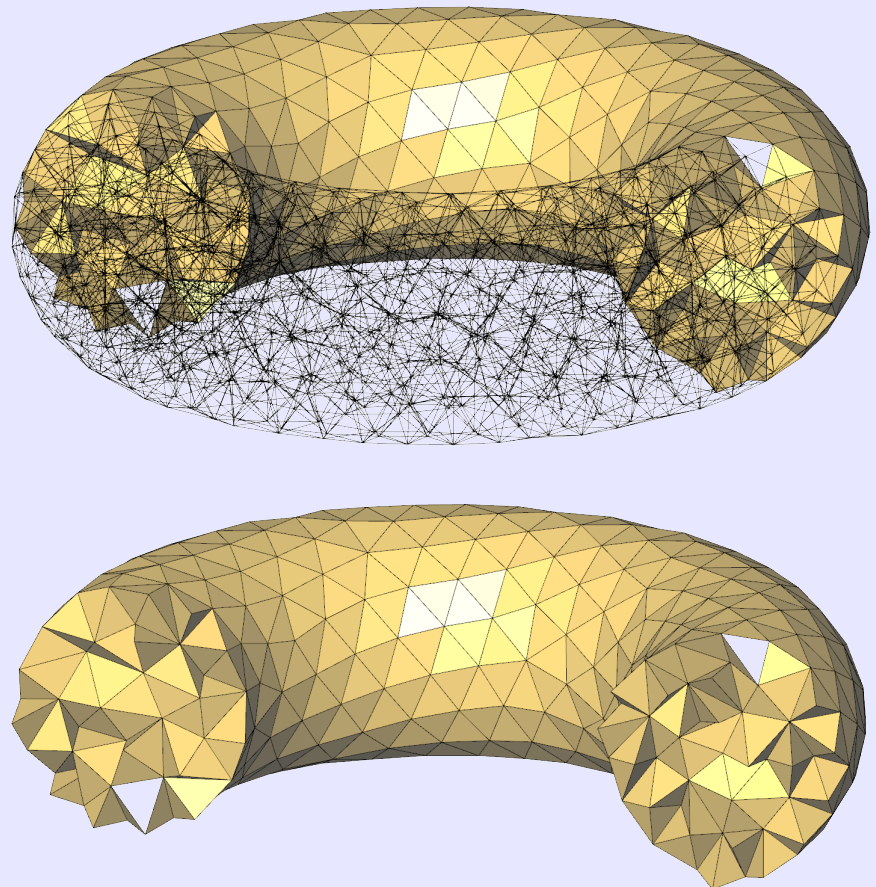
Delaunay
triangulation
restricted to
domain Ω

tetrahedron



circumsphere

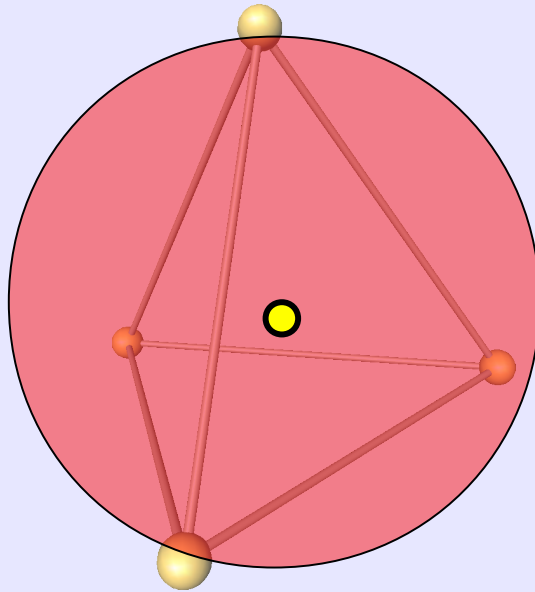
Dual Voronoi vertex
inside domain Ω
("oracle")



Delaunay Refinement

Steiner point 

Bad tetrahedron = big or badly shaped



Volume Mesh Generation Algorithm

repeat

{

pick bad simplex

if(Steiner point encroaches a facet)

 refine facet

else

 refine simplex

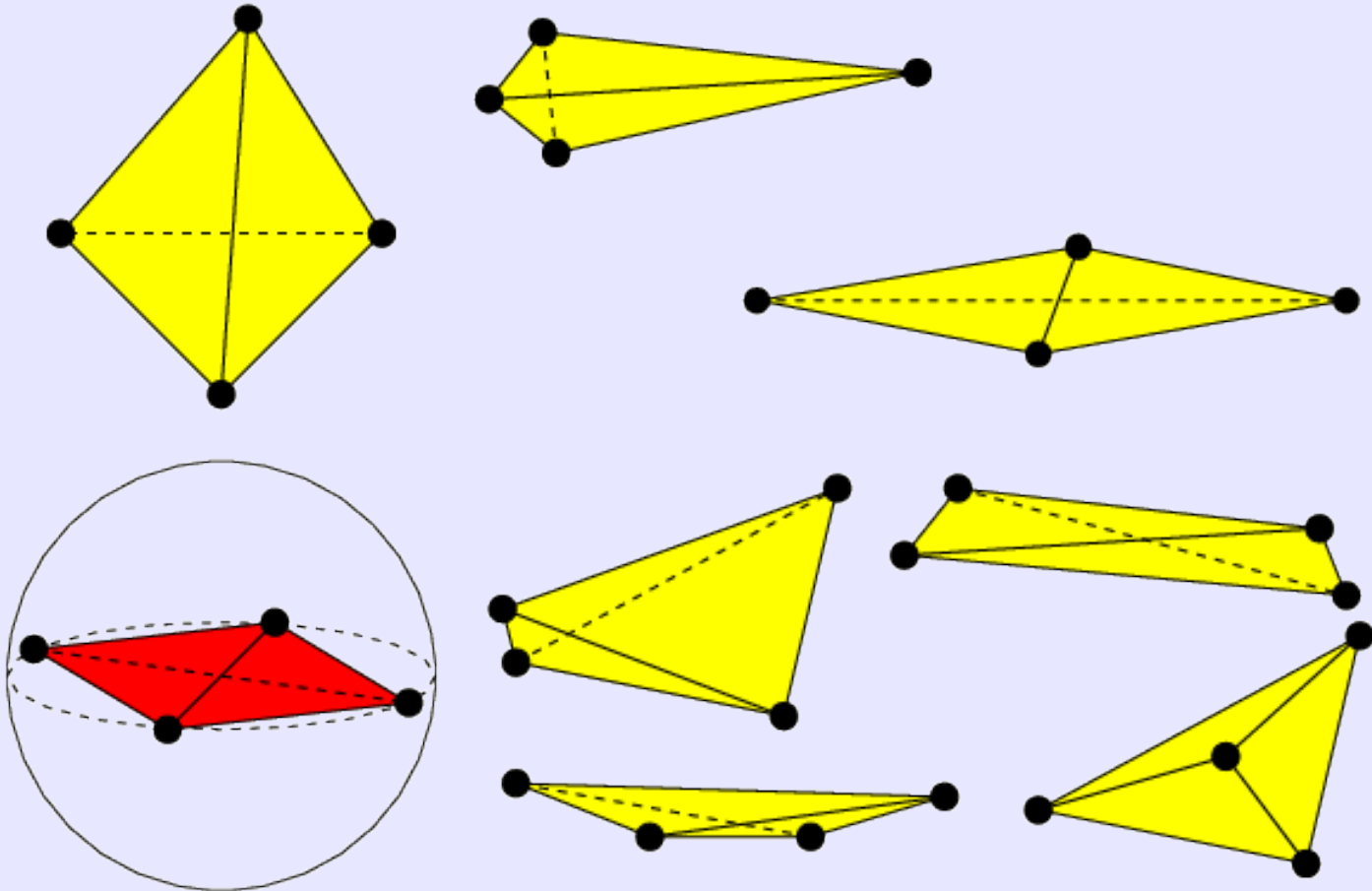
update Delaunay triangulation restricted to domain

}

until all simplices are good

Exude slivers

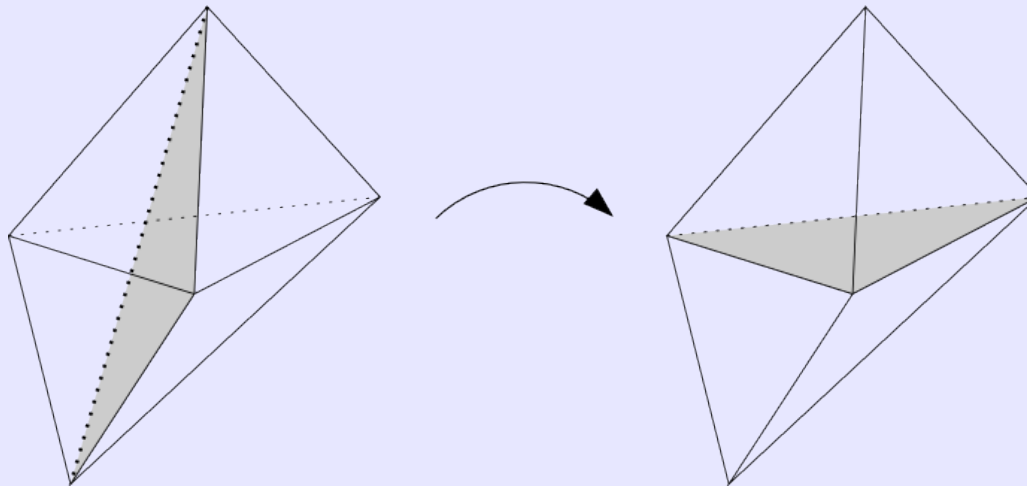
Tetrahedron Zoo



sliver

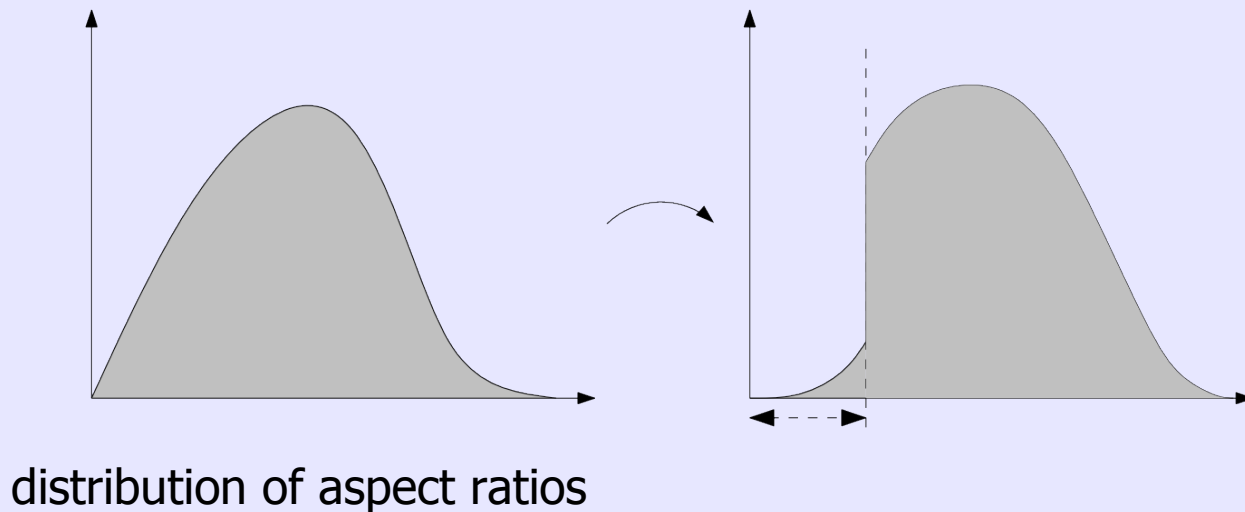
Sliver Exudation [Edelsbrunner-Guoy]

- Delaunay triangulation turned into a regular triangulation with null weights.
- Small increase of weights triggers edge-facets flips to remove slivers.

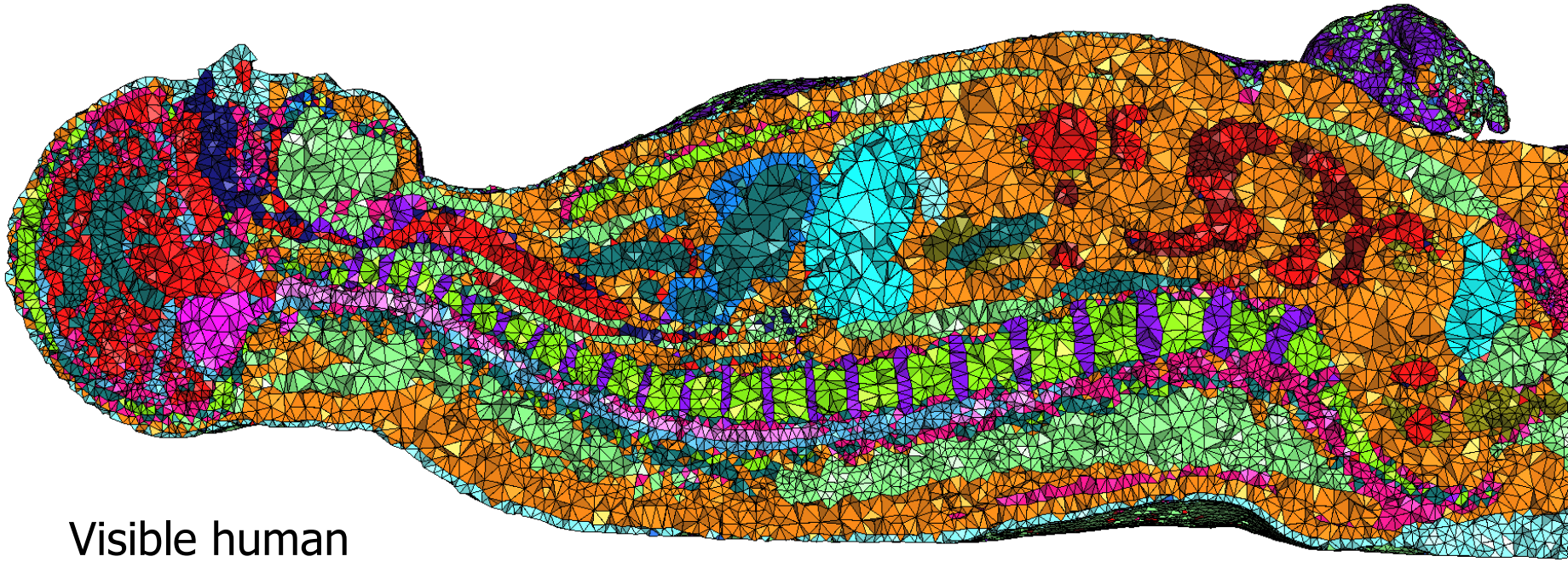


Sliver Exudation Process

- **Try** improving all tetrahedra with an aspect ratio lower than a given bound
- Never flips a boundary facet

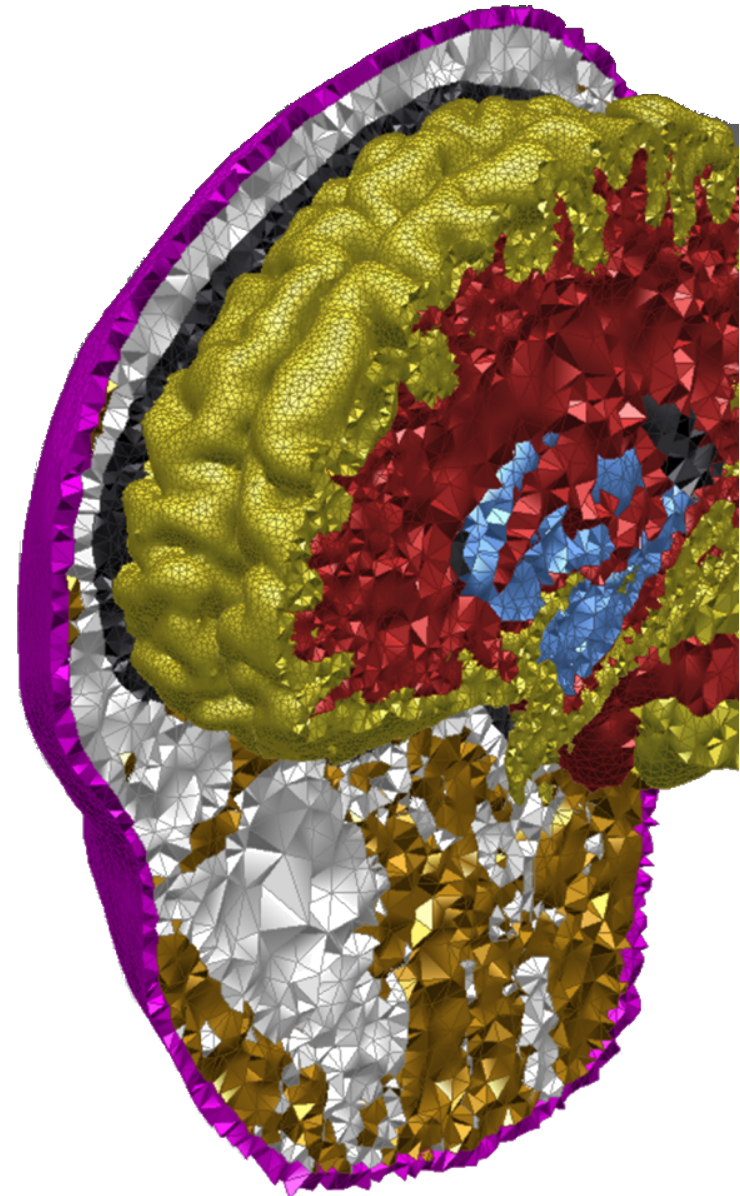
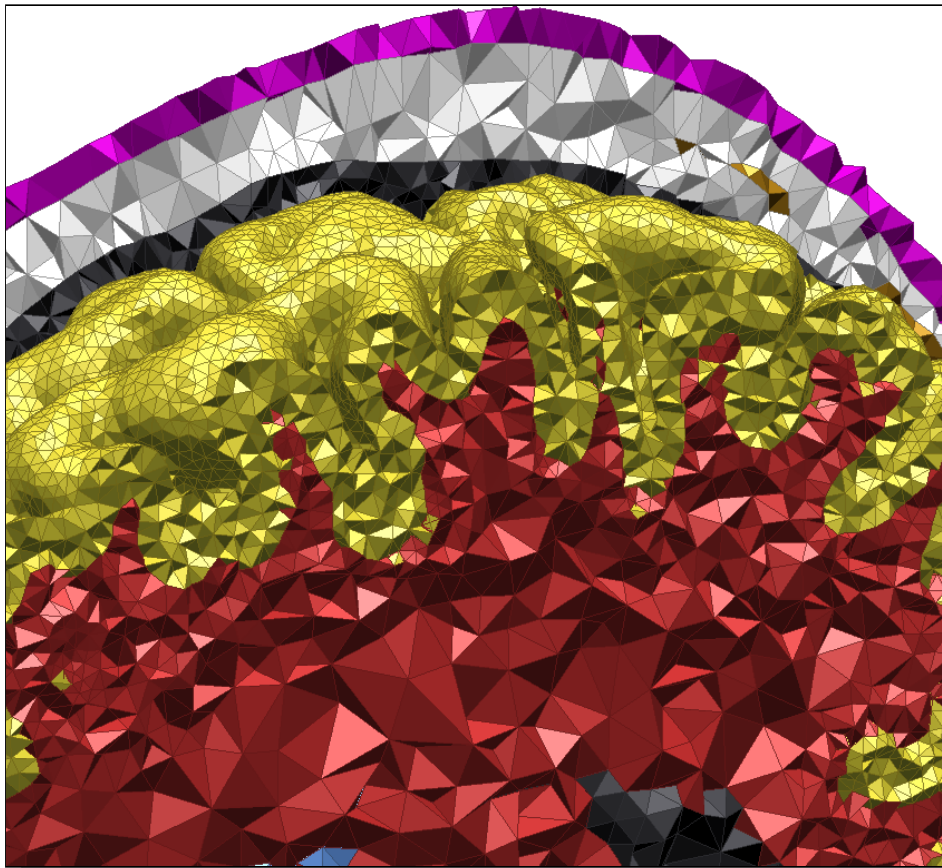


Multi-Domain Volume Mesh



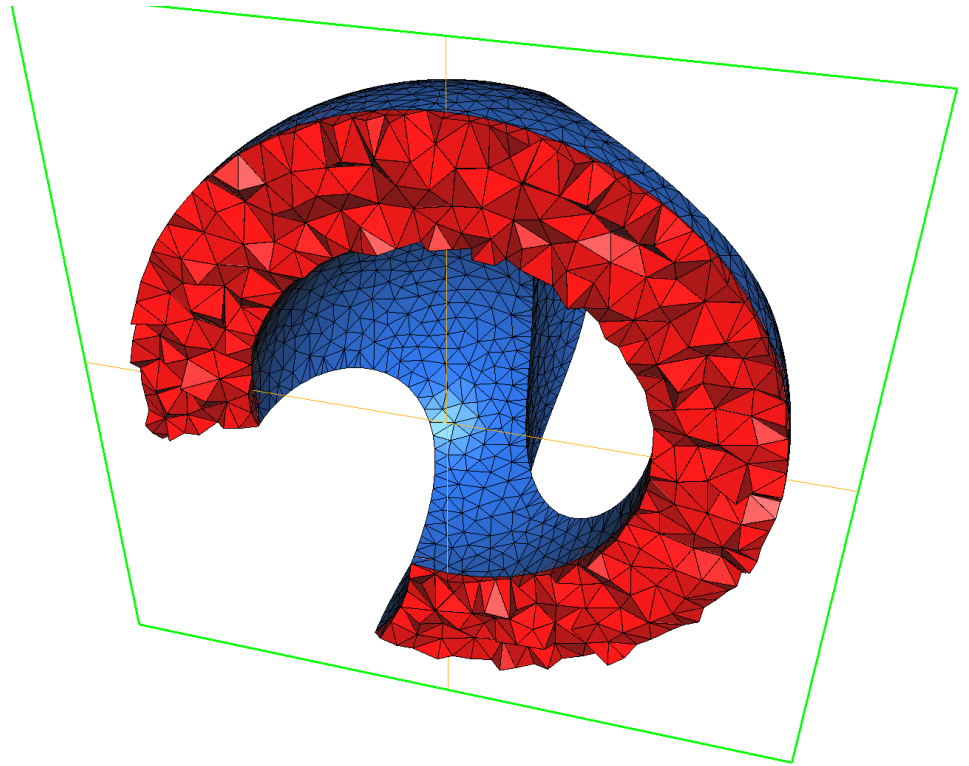
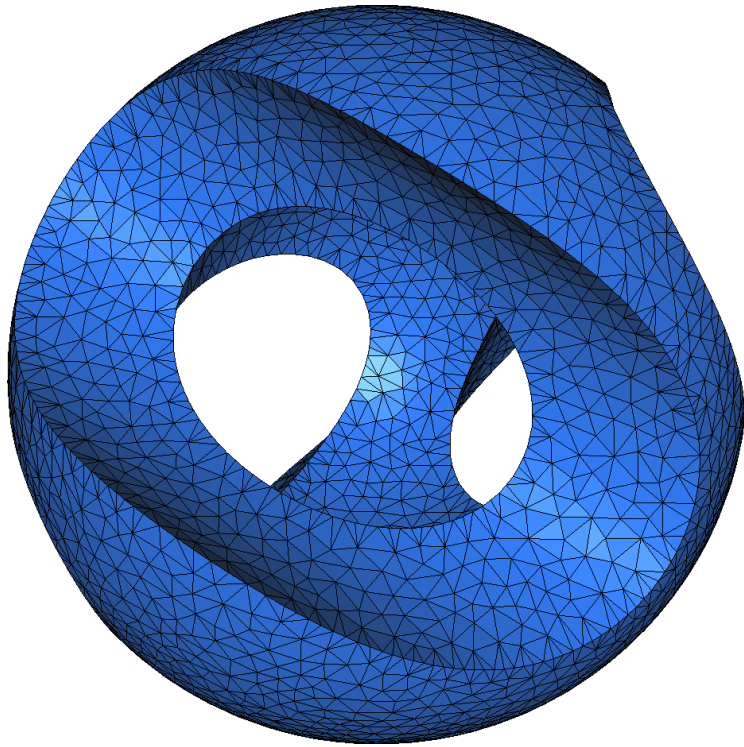
Visible human

Multi-Domain Volume Mesh

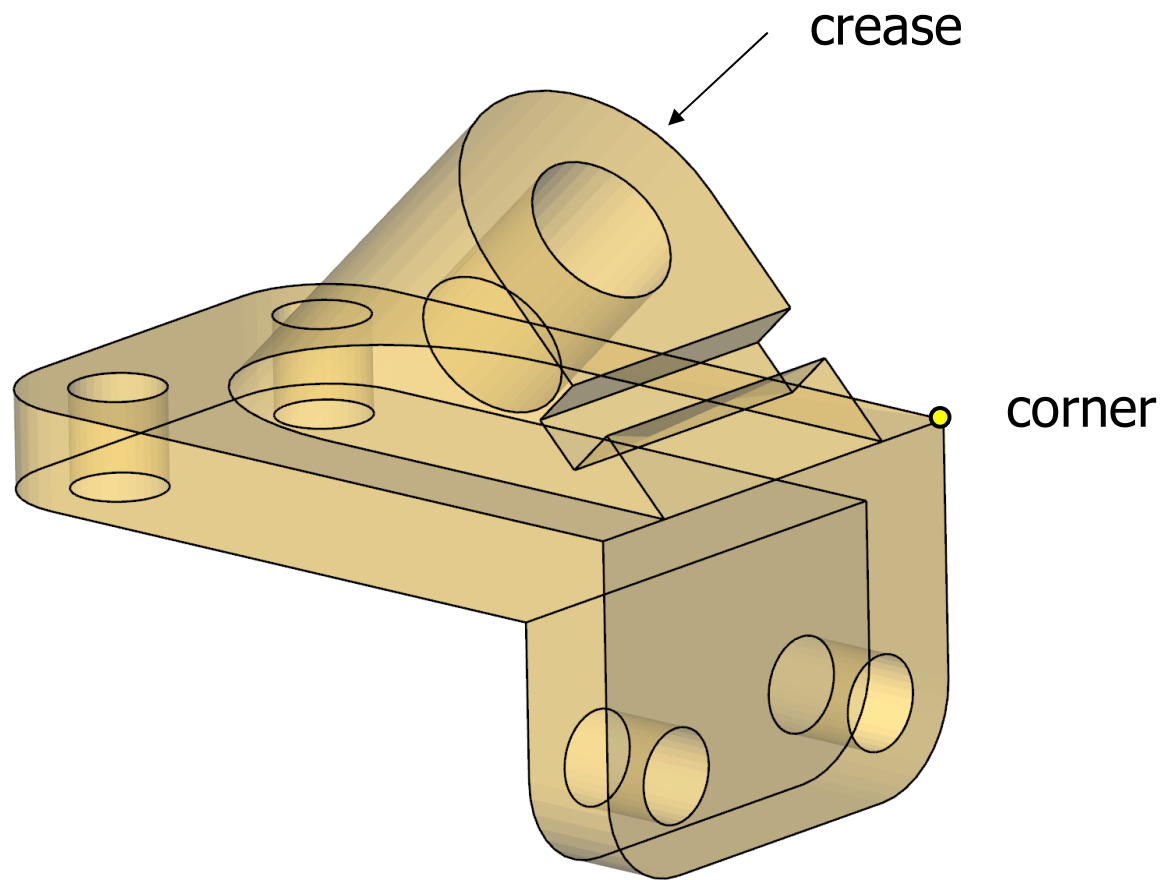


Work in Progress

Piecewise Smooth Surfaces



Input: Piecewise smooth complex



Even More Delaunay Filtering

primitive

Voronoi vertex

Voronoi edge

Voronoi face

dual of

tetrahedron

facet

edge

test

inside

intersect

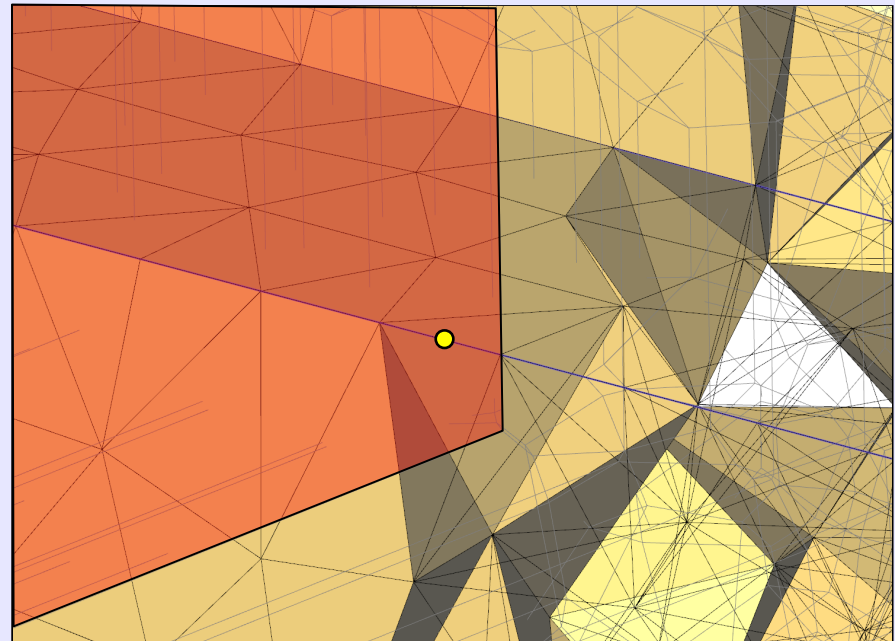
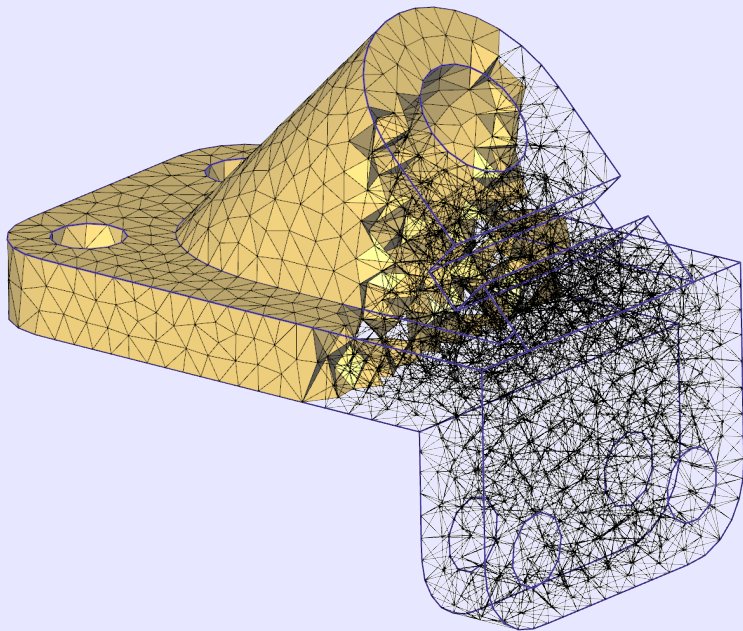
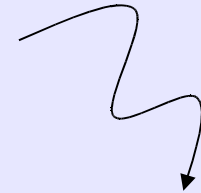
intersect

against

domain

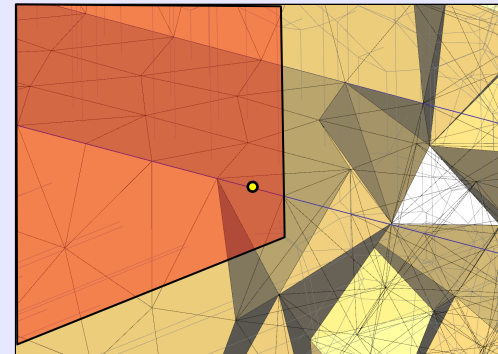
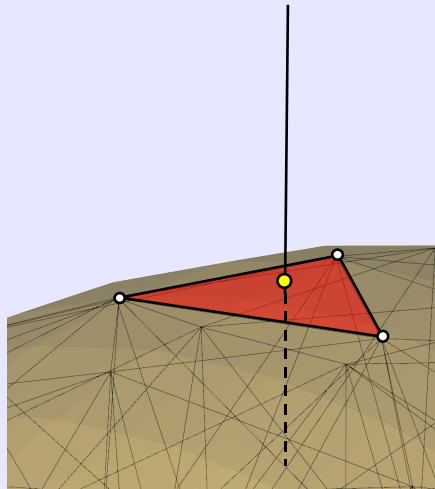
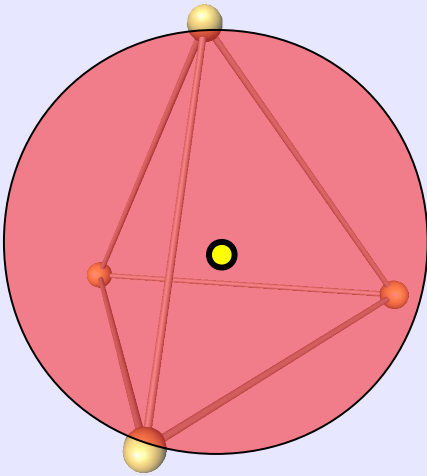
domain boundary

crease



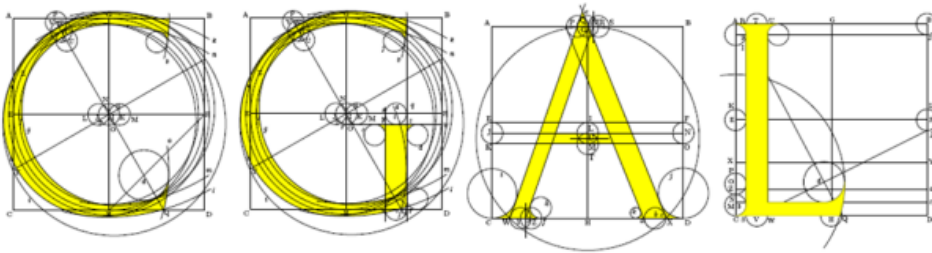
Delaunay Refinement

- Steiner points 



Summary: CGAL for Mesh Generation

- 2D mesh generation
 - From triangulation to quality mesh
 - Preserves constraints exactly
- 3D Mesh generation
 - Interpolates boundary
 - Versatile through oracle-based design



Questions and Answers

Andreas Fabri
GeometryFactory

Pierre Alliez
INRIA

Question and Answers

- General Introduction
- CGAL for 2D Vector Graphics
- CGAL for Point Sets
- CGAL for Modeling and Processing of Polyhedral Surfaces
- CGAL for Mesh Generation