Fair LVC–Curves on Subdivision Surfaces

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1 Introduction

In computer graphics and in computer-aided design one often needs to draw a smooth connecting line between two points on a smooth surface. The most direct such connection is a geodesic line. While it is easy to trace a geodesic ray on a smooth surface or on a finely tessellated polyhedral approximation thereof, it is a well-known hard problem to connect two points with the shortest geodesic path on a surface that exhibits many areas of positive and negative mean curvature.

Sometimes the geodesic line segment is too restrictive for design purposes; it offers no degrees of freedom or adjustable parameters to the designer. This limitation is particularly detrimental when multiple lines must radiate from the same point. In this situation a designer would like to have some control over the initial tangent directions of these lines, perhaps to distribute them at equal angles around the point from which they emerge. For this purpose, a good alternative is a line for which its geodesic curvature is either constant or varies linearly as a function of arc length, like a Clothoid in the plane. Such lines with linearly varying curvature (LVC) offer the designer two parameters: the values of geodesic curvature at either end of the line segment. These can then be used to set the tangent directions at the two end-points (similar to the controls available in a Bézier curve in the plane). We have developed a scheme to efficiently calculate a good approximation to such LVC-curves on subdivision surfaces.

We will illustrate the use of this technique with an example from mathematical topology concerning a crossing-free embedding of a graph on a surface of a suitably high genus. For instance, "K_{12}", the complete (fully connected) graph of 12 nodes, requires a genus-6 surface for an embedding with no crossings, and the 66 edges of this graph will then divide the surface into 44 3-sided regions. To make pleasing-looking, easy-to-understand models of this partitioned surface, we want to make all edges as "fair" as possible, that is, keep them nice and smooth with no unnecessary undulations. At the same time we would like to have the edges more or less evenly distributed around the nodes where they join. LVC-curves offer just the right amount of control for our purpose.

2 Our Approach

The designer starts by constructing a coarse polyhedral model of the needed genus-6 surface as shown in Figure 1a. Choosing oriented tetrahedral symmetry for this surface and exploiting this symmetry to the fullest, the user only has to construct 1/12 of the surface, which can easily be done with 9 quads or 18 triangles. The complete surface is then constructed by composing twelve copies of this fundamental domain with suitable rotations. On this surface, the user now places the nodes of the graph and draws piecewise linear connections between them. If the graph also is given the same tetrahedral symmetry, then this work need be done only on the fundamental domain, i.e. on 1/12th of the surface.

Our algorithm starts from this linear model. The triangle or quad mesh is the basis of a Loop or Catmull-Clark subdivision surface. The polygonal paths between nodes will be converted into LVC-curves. The two refinement processes occur in parallel. For each generation of the subdivision process, each piecewise linear path is modified so as to approximate an LVC curve segment.

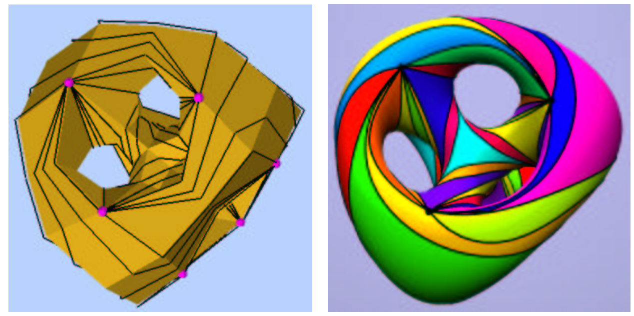

Figure 1. (a) Initial piecewise linear paths on polyhedral model. (b) Final optimized LVC curves on subdivision surface.

Towards this goal, the vertices where the paths cross over the edges of the control mesh are moved with a gradient descent method to approach the desired LVC-behavior (Figure 2). Specifically, each such vertex is moved along the edge on which it lies so as to drive a discretized estimate of geodesic curvature at that point towards the mean of the geodesic curvature values at the two neighboring points on that path. A few dozen iterations of this optimization step are typically sufficient. After this curve optimization process has converged, the surface is subjected to another subdivision step. All linear path segments across any facet in the mesh are then split at the new subdivision edges, and all the path vertices are subjected to the curve optimization process again. This general process loop is repeated until the desired degree of refinement has been reached. The technique works with any of the popular surface subdivision schemes.

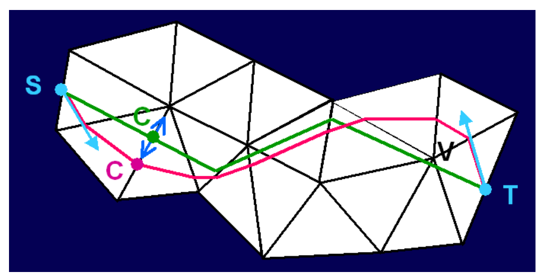

Figure 2. Optimizing a discretized LVC curve linking **S** and **T**; green is the original path, red is the optimized path.

3 Results

The result of this process for the embedding of the K_{12} graph on a genus-6 surface of tetrahedral symmetry is shown in Figure 1b. The LVC curves have been enhanced to black bands to make them more visible, and the nodes of the graph are shown as small hemispheres. The 44 resulting 3-sided facets between the edges have been colored randomly. Thus we are able to provide a crisp visualization model for this difficult graph-embedding problem.

For a computer-graphics audience, the colored facets can be made of translucent material, and light bulbs can be placed inside the four tetrahedral corners as well as the center of this object, so that an intriguing looking "Tiffany Lamp" of genus 6 results. Stunning visual effects are achieved by subjecting this geometry to various computer graphics rendering techniques.

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