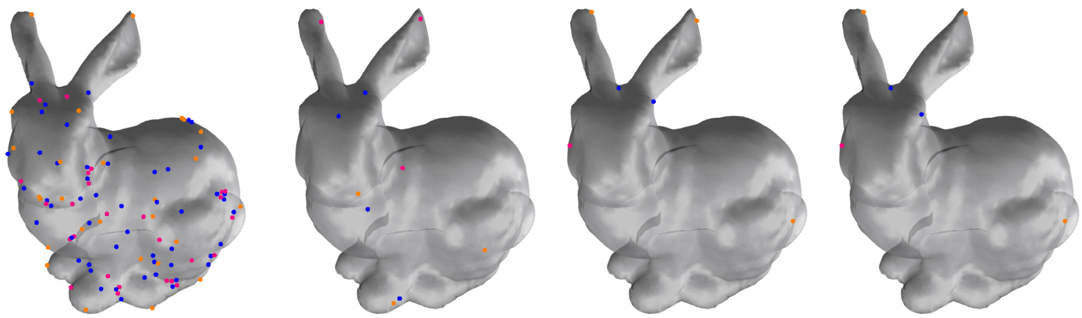
Critical Points with Discrete Morse Theory

Peihong Guo & Ergun Akleman
Departments of Visualization &
Computer Science and Engineering
Texas A&M University

He Ying & Xiaoning Wang
School of Computer Engineering
Nanyang Technological University

Wei Liu
School of Energy and
Power Engineering,
Jiangsu University



(a) Essential Vertex Defects 53 Max/Min & 51 Saddle | (b) Linear Function 6 Max/Min & 4 Saddle | (c) Geodesic Function 4 Max/Min & 2 Saddle | (d) Random Function 4 Max/Min & 2 Saddle

Figure 1: *Examples of critical points: red points are maximum/minimum and blue points are saddles.*

In this work, we present some of the unexpected observations resulted from our recent research. We, recently, needed to identify a small number of important critical points, i.e. minimum, maximum and saddle points, on a given manifold mesh surface. All critical points on a manifold triangular mesh can be identified using discrete Gaussian curvature, which is given as $\kappa_i = 2\pi - \sum_j \theta_{i,j}$ where κ_i is vertex defect (the discrete Gaussian curvature) of the vertex i and $\theta_{i,j}$ is the corner of the vertex in the triangle j. A very useful property coming with vertex defect is the discrete version of Gauss-Bonnet theorem: the sum of all vertex defects is always constant as $\sum_i \kappa_i = 2\pi(2-2g)$ where g is the genus of the mesh. Any vertex with a non-zero vertex defect is really an critical point of the surface. However, identification of interesting critical points is hard with vertex defect alone. As it can be seen in Figure 1(a), even we ignore vertex defects that are small, too many vertices are still chosen and this information is not really useful to make any conclusion of the shape of the surface.

Morse theory provides an alternative to use Gaussian curvature. Smooth Morse theory states for a surface that is derivative continuous everywhere, using the gradient of a derivative continuous function $F : \Re^3 \longrightarrow \Re$, we can classify critical points as equilibrium points of ∇f on the surface. Similar to Gaussian curvature, Morse theory shows that if we assign a winding number ω_m to all critical points, $+1$ to minimum and maximum points, -1 to simple saddles, $-k$ to complicated saddles where k a positive integer; then $\sum_m \omega_k = 2-2g$, which is the Euler characteristics of the surface. Although, smooth Morse theory does not formally support piecewise linear case, it is still used for identify important critical points on triangular meshes. If f is a linear function in the form of $f = ax+by+cz$, it is not unexpected that Morse theory can hold on triangular meshes. However, in practice the number of critical points again comes more than desired. Therefore, we apply a smoothing operation to reduce the number of critical points. However, even smoothing does not reduce number of points significantly (see Figures 1(b)).

Another alternative to identify critical points is Discrete Morse theory. In discrete version of Morse theory, we simply use discrete gradient fields, which are the gradients of some discrete functions that are defined only on mesh vertices. These are really discrete conservative vector fields which can be represented as directed graphs embedded to surfaces with no cycles. We, first, decided to try Geodesic function to identify discrete Morse points. Since a geodesic function is not defined in 3D and its derivative is not necessarily continuous on a triangular mesh, smooth Morse theory cannot theoretically apply geodesics. Therefore, Geodesics functions are perfect candidates for discrete theory. We have confirmed that the sum of the winding numbers of critical points is still equal to $2-2g$. To our surprise, a few iteration of smoothness was sufficient to obtain very nice locations for critical points and the number of critical points from geodesic function was fewer than the number from linear function (see Figures 1(c)). More interestingly, we observe that even assigning random numbers to each vertex provides good results. Interestingly, the results were almost identical to geodesic after few number of mesh smoothing operations applied to random data (see Figures 1(d). This particular observation suggests that in many cases combinatorial structure of the meshes could be more important than vertex positions.

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