

Computer Aided Sculpting

A Multidisciplinary Course in Shape Modeling

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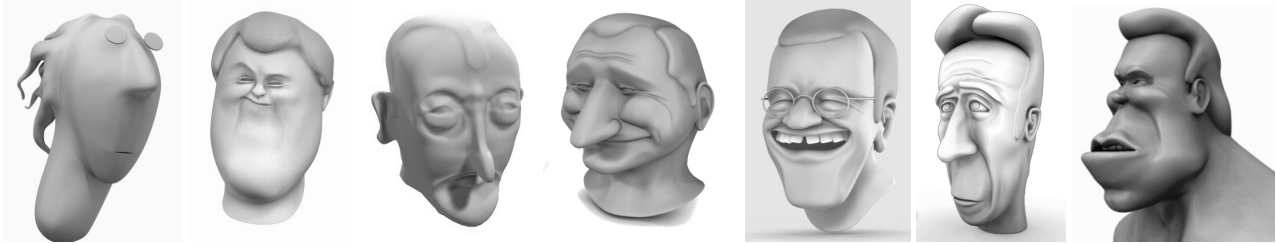


Figure 1. Examples of Expressive 3D Caricatures Created in *Computer Aided Sculpting* Course

Abstract

Computer Aided Sculpting course is an experimental interdisciplinary course that focuses on both artistic and scientific aspects of shape modeling. I have been teaching this course every year since 1996. The problems coming out of the course influenced the development of our Topological Mesh Modeling system, called TopMod. More than 10 students added new features to TopMod while they were taking Computer Aided Sculpting course as their projects. I have at least ten published papers and sketches about the techniques I have developed while I was teaching the course.

See <http://www-viz.tamu.edu/faculty/ergun/teaching/index.html> to go to class web-pages.

1 Introduction

I am currently program coordinator of Master of Sciences in Visualization Sciences program in Department of Architecture at Texas A&M University. As an educator with both scientific and artistic expertise, my main aspiration is to successfully integrate science, art and technology in education as well as research. Computer Aided Sculpting is one of my experimental courses that shows science and art can successfully be integrated in one course.

Such an integration requires the teacher to have intuitive and rigorous knowledge in mathematics, theoretical and practical knowledge in computer science, and the skills and talent of an artist. To solve this problem, in some schools art and science faculty members try to co-teach such interdisciplinary courses. However, it is better to find faculty members who can easily switch between their right and left brains. Leonardo Da Vinci is the most well-known left-right brain person. Conventional wisdom says that such people are rare. However, they are much more common than what we normally think. In mathematics and scientific community, I have known many people who can use both sides of their brains. For instance, there are contemporary mathematician/sculptors such as George Hart [Hart 2005], Helaman Ferguson [Ferguson 2005; Fer-

guson et al. 1992], and Carlo Séquin [Sequin 2005] who successfully combine art and mathematics.

I also have both scientific and artistic expertise. I have worked as a professional cartoonist since high school. I am also an engineer who earned his Ph.D. in Computer Graphics from the Department of Electrical and Computer Engineering. During the past ten years, my multidisciplinary direction has presented me with many opportunities, challenges and hurdles, requiring me to develop new research directions and new teaching approaches. In turn, my work in these areas has allowed me to make significant contributions to both the fields of shape modeling and visual storytelling and to the widely recognized success of the Master of Science in Visualization Sciences program at Texas A&M.

Our program is designed to prepare students for a range of long-term careers in computer graphics and visualization. The core curriculum of the MS in Visualization Sciences program is designed to give all students a basic grasp of the artistic, scientific, cognitive and technical foundations of the discipline. Beyond this broad training, the program requires students to develop a strong focus area of advanced expertise and to complete a research thesis in this area.

The program centers on the computer as a primary tool for visualization with strong offerings in animation, modeling and simulation, image generation and manipulation, and supporting technology. Our students come from architecture, computer science, mathematics, engineering, and art backgrounds. They demonstrate strong artistic and technical skills. The program typically has about 45 active students at any given time. Since 1990, more than 150 students are educated by the program, and among 150 former students more than 100 are currently working in animation and special effect industry including PIXAR (20), ILM(10), Dreamworks (10) and Electronic Arts (7). Blue Sky (6), Rhythm and Hues (6), Will Wilton (6). Because of strong influence from the industry and professional nature of our program, I, as a faculty, have to carefully balance theory with practice as well as art with science.

2 My Philosophy of Teaching

My goal in education is to identify and teach fundamental concepts and skills such that our graduates can continuously adjust them-

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†Thanks to all students who take Computer Aided Sculpting course.

selves to the rapid change inherent in the fields of computer graphics, architecture and digital art. I encourage learning by doing and exploring. My courses are organized around various projects that are designed to make the students confront the key issues and problems in the field. I cultivate a sense of friendly competition among the students and I encourage each student to aim higher and to work harder at reaching their full potential. I try to make the classes as *interactive and adaptive* as possible. I combine architectural studio approach with lectures. In my courses, students simultaneously learn art, mathematics and computer science.

The studio approach requires individual interaction and critiques along with lectures. In teaching studio courses I introduce methods used in science and engineering education, identifying fundamental concepts and skills and developing well-defined projects. I use individual and group critiques to evaluate the projects. I use these critique sessions to teach students to be comfortable with critiques, to encourage them to continuously improve their work and to develop good work ethics. An indication of the success of my overall approach is that the quality of the student work in my studio courses is continuously improving.

3 Motivation for Computer Aided Sculpting Course

With the advance of computer graphics, many artists have also begun to use mathematics as a tool to create revolutionary forms of sculptures. There are many contemporary sculptors such as George Hart [Hart 2005], Helaman Ferguson [Ferguson 2005; Ferguson et al. 1992], Bathseba Grossman [Grossman 2005], Brent Collins, and Carlo Séquin [Sequin 2005] who successfully combine art and mathematics to create unusual and high-genus sculptures. These mathematical sculptors, who have a very noticeable presence in today's art scene, develop their own methods to model, prototype and fabricate an extraordinary variety of shapes. Most of these contemporary sculptors who successfully combine art and mathematics are, in fact, mathematicians or engineers who have developed their own methods.

One of the most exciting aspects of sculpting has always been the development of new methods to design and construct unusual, interesting, and aesthetically pleasing shapes. One of the most interesting sculptural shapes are high-genus surfaces [Takahashi et al. 1997; Welch and Witkin 1994; Ferguson et al. 1992; Srinivasan et al. 2002; Akleman et al. 2003]. My goal in computer aided sculpting course is to develop and teach reproducible procedures to construct interesting sculptures. I also want to provide creative flexibility to my students for constructing a wide variety of sculptures that can still be recognizable as belonging the same family. I want the resulting shapes to be physically realizable. In other words, students should be able to 3D print the resulting shapes.

4 Course History and Future

Since 1996, I regularly teach *Viza 657, Computer Aided Sculpting*. An important element missing from our visualization program was a course covering scientific and artistic aspects of shape modeling, which is a subject central to the fields of computer graphics and architecture and I developed and offered the *Computer Aided Sculpting* course two times as an experimental course, and in fall 1998 the course became a regular offering. The course was cross-listed with Computer Science in 2000 and continuously attract students from engineering and science programs. The

course web-pages starting from 1998 is available at <http://www-viz.tamu.edu/courses/viza657/index.html> page.

Opening a new course or changing course content is difficult in our program. Since the main goal of our students is to find technical directing jobs in special effects and animation industry, they try to develop a strong portfolio that will help them find jobs. Therefore, they try to avoid any course that does not seem to serve this purpose. They do not even like any change in course structure if the change does not seem directly related to their goal. Despite this hurdle I have created 5 new courses during my tenure in Texas A&M. Computer Aided Sculpting has probably most interesting history in terms of student interests. During the first few years, course was able to offered just because of graduate students of computer science, engineering and mathematics. None of our own students took the course during this time period.

Visualization students suddenly discovered the course after several years. Then, a significant majority started to take the course. Some even complained from non-visualization students taking the course and using our own resources, partly my time. Last year, I moved the course toward non-representational sculptures and 3D printing. Since in movie industry actual construction of the shapes is not an issue, our own students did not like the change. But, architecture students now love the course. As a result of this change, I expect architecture and computer science students will again start to populate the class. After a few years, when visualization students eventually learn the benefits of the course for their job search, they will again complain non-visualization students taking the course. It seems like a never-ending cycle.

In Computer Aided Sculpting course, lectures that cover the scientific and mathematical aspects of computer graphics are still essential. I use weekly quizzes to identify any weakness in understanding of topics. However, my main contributions in teaching computer aided sculpting are (1) projects which are used the main vehicles for teaching concepts and skills; and (2) evaluation of project using the methods used in studio courses to integrate artistic sensibility. For the reminder of the paper, I will focus on projects and their evaluations.

5 Projects

One of the ultimate goals of computer graphics is to develop techniques to create wide variety of artworks such as drawings, paintings, sculptures and animations. Development of these techniques requires a good understanding of abstraction, simplification and exaggeration.

The concepts of abstraction, simplification and exaggeration are essential parts of visual arts. These concepts are always employed even in creation of drawings, paintings, sculptures and animations. The close examination of very realistic looking artworks reveals that abstraction, simplification and exaggeration are widely used in creation of even such realistic works [2].

Fine artists always ignore unnecessary details and focus on the characteristic features of their subjects. For instance, no classically trained painter will draw every visible detail in a still life. Caricaturists not only ignore unimportant details, but also selectively exaggerate the features that makes their subjects unique.

The goal of my projects is to teach abstraction, simplification and exaggeration with hand-on experiences. I can organize my projects as representational sculptures and non-representational sculptures.

5.1 Representational Sculptures

As representational sculptures, I had a wide variety of projects such as figure sculpting and plant modeling. However, most original type of representational sculpture projects are 3D facial caricatures. In this section, I will focus on these expressive caricature projects.

Of course, one reason I focus on caricatures is because I am a cartoonist and caricaturist. However, that is not only reason. Although abstraction, simplification and exaggeration are very common tools used in visual arts, only in caricature we consciously learn to apply them. Unfortunately, caricature is one of the orphan fields in fine arts. In United States, caricature is not considered high level of art and disregarded in academic circles. Even public view caricature as a consumer product; Caricature is consumed and forgotten.

I have seen and reviewed many caricature papers that attempts to automatically exaggerate from frontal face images and almost all of them were unsuccessful. I am not saying this as a computer scientist but as a caricaturist. In fact, I also failed miserably in terms of my prediction on how easy it is to identify the features to be exaggerated. In 1997, I had a Siggraph sketch [Akleman 1997]. In my presentation, I claimed that anybody can identify what to be exaggerated by using morphing. Identification of the unique features is essential for creating caricatures since those features are the ones that will eventually be exaggerated. The procedure based on the image morphing that consisted of five stages and it was really simple.

I also had a paper on morphing that shows extreme exaggerations with deformation using simplicial complexes. One graduate student developed a thesis on caricature interface. But, the fact that I was still the only one who can do caricature using these interfaces.

5.1.1 Procedure to make 3D Caricatures

Based on my experience and claim, I slowly made caricature as a part of the curriculum of Computer Aided Sculpting. Unfortunately, it never worked unless I am heavily involved with the process. (Our students are not artistically untalented programmers, We choose our graduate students partly based on their portfolio. They are coming from a wide variety of background such as CS, Architecture and Art but all have a basic art talent and education. Despite that they were not able to do it.)

The last two years, I spent much more time for caricature homework. I worked with each student. I draw them sketches. I made them to create shapes using disconnected NURBS models such that I can make changes easily. Eventually they converted models to subdivision. This time, almost all were successful. Some examples are shown Figure 1 and 2. Based on all this experience, I have developed the following procedure to create 3D caricatures. The procedure consists of four stages:

1. **Data collection.** In this stage, students choose a well known person with an easily recognizable face. For caricatures, we usually use movie stars. In addition to being easily recognizable, the faces of most movie stars include some unique features that help creating caricatures. During this stage, students collect photographs and caricatures of the person from a wide variety of sources such as movie clips and magazines. If it exists, 3D scanned models and video clips can also be useful.
2. **Unique (Exaggerated) Feature Identification.** A feature is called unique if it is different than average. Identification of the unique features is essential for creating caricatures since

those features are the ones that will eventually be exaggerated. Students identify the unique features of a given face by using the following procedure [Akleman and Reisch 2004; Akleman 2006]. The procedure based on the image morphing consists of five stages.

- (a) Start with a representative image of the person.
- (b) Create an very simple template for image morphing.
- (c) Exaggerate only one feature at a time.
- (d) If exaggeration creates a likeness, continue to exaggerate. If it does not create a likeness, make the exaggeration in the opposite direction. If neither direction gives a likeness, return the feature to its original position.
- (e) Continue with another feature until all unique features are identified.

Note that it is impossible to identify features from just a single photograph since a face is changing dynamically. For instance, our mouth moves up when we smile. If we use a frontal picture of a person smiling and disregard the fact that she is smiling, we may mistakenly identify her chin as a long chin. We may think that the distance between his nose and lips is short. Collecting large number of images during data collection stage helps to reduce wrong feature identification.

Having 3D scanned models can also be useful for feature identification but it is still not enough since it capture only static face. To identify the unique features we need to capture a dynamic 3D face by combining video with 3D scanning. Moreover, some features can be almost un-measurable. A good example is George W. Bush's face. If you look at his neutral photo, it is almost impossible to find out his right and left side is not symmetric. But, I know this fact from his father's face and only when I include this asymmetry to his face his caricature become recognizable.

Caricature is really like science. Each person is an unknown to be discovered. Each caricature of the person is like a science paper that provides us another information about the person. The caricaturists collectively discover the truth. We, caricaturists, have our Newtons or Einsteins like Kruger or Piven, but most of us are like average talented scientists. We learn from each other. We perfect each other. Caricaturing is a collective process. You can see this collective process is in action as soon as a new president is elected. For instance, George W. Bush's eyes are smaller than normal. But, the caricaturists did not discover it as soon as he was elected president. But, after six months, every caricaturist was able to draw a good likeness.

3. **Abstract Caricature Creation.** In this stage, students create abstract caricatures using disconnected pieces. Using disconnected pieces is partly motivated by cubist sculptures such as Pablo Picasso's Reclining Bather. Individual pieces also allow faster shape modification. Each unique feature is represented by at least one disconnected 3D surface. Therefore, it is easy to improve 3D recognizability of the caricature shapes by changing the position, shape and size of each feature. Figures 2 and 3 show two examples of 3D caricature created with disconnected pieces. These examples show two completely different approaches to create abstract caricatures. In Sylvester Stallone's caricature, all the detailed features are modeled. On the other hand, in the Julia Roberts case, only a few unique features are modeled and an important feature (hair) is completely omitted. Despite the differences, both provided likeness, which is useful for the next stage. As a side note: In both cases, disconnected pieces are modeled by using



Figure 1: More Examples of Expressive 3D Caricatures Created in *Computer Aided Sculpting* Course.

NURBS, but this is not a necessity. Subdivision surfaces can also be used to model individual pieces.

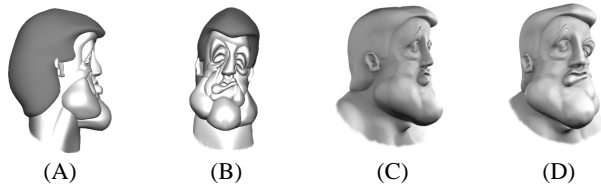


Figure 2: Sylvester Stallone by Jacob Brooks. (A) and (B) show modeling with disconnected pieces. (C) and (D) show final 3D caricature.

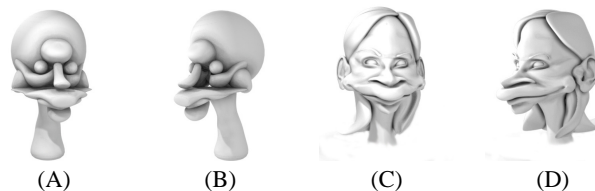


Figure 3: Julia Roberts by Jon Reisch: (A) and (B) show modeling with disconnected pieces. (C) and (D) show final 3D caricature. (This particular facial expression is motivated from her performance in *Notting Hill*, when she said "I am just a girl standing in front of a boy asking him to love her".)

4. **Final Modeling and Rendering.** Once the shape that is constructed with disconnected pieces is confirmed to be a likeness of the person, the students create a subdivision surface that closely approximates the confirmed shape. Figures 2 and 3 show two examples of 3D caricatures created with subdivision surfaces.

To measure the success of the 3D caricature, we have to be very careful. People's tolerance is so high that it is easy to make people accept unsuccessful caricatures as successful ones. There are several tricks of trade, we need to be careful to avoid. Thus, we use an extremely simple rendering approach. We just use an Ambient Occlusion shader that reinforces areas of subtle relief on the models and provides a clean, consistent look to the final images. By omitting the use of textures and color, we ensure that the shapes alone provide the recognizability of the caricature.

I have very simple litmus test to measure the success of caricature. If caricature is good, the original image should not look like the person. In other words, the caricature must blow out our common sense. Although, we know that which one is real, we should still feel that the caricature is better likeness.

5.2 Non-representational Sculptures

A significant portion of projects in Computer Aided Sculpting course has become non-representational type. I have developed projects such as nested sculptures, interlocked sculptures, fractal shapes, shapes in shapes, developable surfaces, dynamic sculptures. Among all projects, the most representative of non-representational sculptures is connected & symmetric high genus sculptures.

5.2.1 Procedure to model Connected & Symmetric High Genus Sculptures

For this project, I first teach a procedure that illustrate several aspects of topological mesh modeling. This procedure is based on a set of topological mesh modeling operators. These operators guarantee that the resulting shape is topologically 2-manifold. If the user avoid self-intersection, which is easy to achieve, the resulting models are 3D-printer-ready, i.e. they can be printed by using a rapid prototyping machine as shown above. The procedure consists of six stages.

- **Stage 0: Initial Shape.** Start with a symmetric convex polyhedral shape [Cromwell 1997; Grunbaum and Shephard 1987; Wells 1991]. Any platonic solid such as tetrahedron, cube or dodecahedron [Stewart 1991; Williams 1972] are perfect candidates for starting shape. In the example shown in Figure 4, the initial shape is a cube.
- **Stage 1: Extrusions.** Apply the same extrusion operation to all faces of starting shape. Adding a twist or rotation to the extrusions can create an additional effect as shown in Figure 4. Moreover, the recently introduced local mesh operators can also be used in this stage. These local operators can extrude generalized platonic solids and are called tetrahedral, cubical, octahedral, dodecahedral and icosahedral extrusions [Landreanu et al. 2005].
- **Stage 2: Creating Handles.** Connect symmetrically related faces using multi-segment curved handles [Srinivasan et al. 2002]. This operations, along with cubical extrusions, populate the mesh with quadrilateral faces with 4-valent vertices as shown in Figure 4. After the operation, there will be only a handful of extraordinary points (i.e. vertices with valence other than 4 and faces with sides other than 4).
- **Stage 3: Doo-Sabin Smoothing.** Apply Doo-Sabin subdivision once [Doo and Sabin 1978]. This operation will create visibly connected clusters of faces. These cluster of faces are formed as a result of extraordinary points. The faces in each cluster defines a route that connect one extraordinary point into another. These clusters can easily be seen in Figure 4.
- **Stage 4: Rind Modeling.** Rind modeling creates a rind structure by creating a smaller replica of initial shape [Akleman

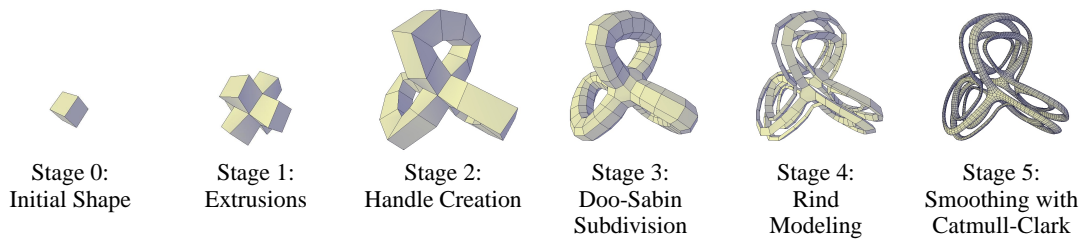


Figure 4: The Procedure.

et al. 2003]. In rind modeling, users can punch holes in this ring shape by clicking the faces. There exist two strategies: (1) Deleting all non-cluster faces by making the clusters clearly visible, (2) Deleting all cluster faces by making the non-cluster structures visible. In the example shown in Figure 4 we made clusters visible.

- **Stage 5: Final Smoothing.** Final smoothing is useful to create a simple and smooth surface. Although, in final smoothing any subdivision scheme can be used, we use Doo-Sabin [Doo and Sabin 1978] or Catmull-Clark [Catmull and Clark 1978] schemes for smoothing.

Using both stages 2 and 3 is not really required. It is even possible to skip either one of them to create different results. It is possible to create a wide variety of shapes using this procedure as shown in Figure 5.

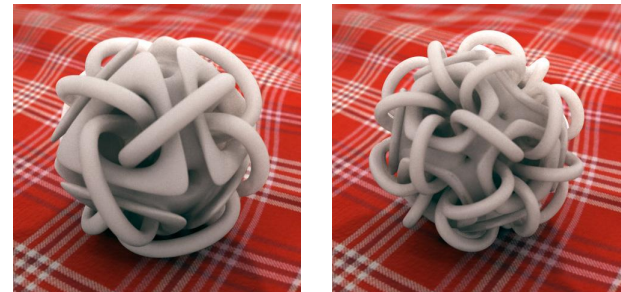
5.2.2 Creative Flexibility

I observe that, using the procedure, students can rapidly create a wide variety of shapes. Although these shapes are completely different; they indistinguishably belong the same family. Moreover, The resulting shapes can be physically realizable with a minimal effort from user as shown in Figure 5. Figure 6 shows some examples of shapes that were created by some of the students using our procedure, as one of the biweekly projects of computer aided sculpting course.

Figures 7 and 8 show a student's (Cem Yuksel) works that illustrate creative flexibility that is provided by the procedure. The figure shows the steps that is used by the student and final virtual sculptures. Cem by adding handles after completing the basic procedure created two forms that challenge with each other.

5.2.3 Software

The operations used all six stages have already been implemented and included in our existing 2-manifold mesh modeling system, called TopMod [Akleman and Chen 1999; E. Akleman and Srinivasan 2003; Akleman et al. 2003; Srinivasan et al. 2002]. Our system is implemented in C++ and OpenGL. You can download TopMod from <http://www-viz.tamu.edu/faculty/ergun/research/topology/> for and use the software for non-commercial applications. TopMod provides only Open-GL based interactive rendering. For high quality rendering, shapes can be exported in obj format and rendered in some 3D modeling and animation system. Our students use Maya and 3D Studio Max for rendering the final models.



Rendered Model with Subsurface Scattering

Another Model created with the same procedure starting from Icosahedron

Figure 8: Cem Yuksel's final virtual sculptures. Both rendered using subsurface scattering to give an illusion of ABS plastic.

5.3 Conclusion and Future Work

Sculpting and Architecture can provide new directions for teaching artistic nature of solid and shape modeling. In both, the precision, which is very important for engineering shape design, is not a major concern. Both architecture and sculpture can provide us interesting educational problems coming from aesthetics concerns. Identification of methods to create new aesthetically pleasing sculptural and architectural forms can eventually be a major direction for solid and shape modeling research.

In this work, we have introduced a course to teach shape modeling with a motivation coming from strong aesthetics concerns. My techniques are reproducible and allow educators to create their own techniques to teach shape modeling.

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Figure 5: Models created in TopMod and 3D prints of the models For each sculpture, we took two photographs from slightly different point of views. The sculptures are photographed on a mirror. Background is eliminated. These shapes are made from ABS plastic and printed using a Fused Deposition Machine (FDM). They are later painted using an acrylic paint. The ribbon shape on the left is really a genus-1 surface, but, during the construction of this surface, genus changed several times.



Figure 6: Rendered examples of high genus symmetric sculptures designed by students who take a computer aided sculpture course. Although, these images are created by seven different students; shapes and rendering styles are completely different, all the virtual sculptures are clearly from the same family.

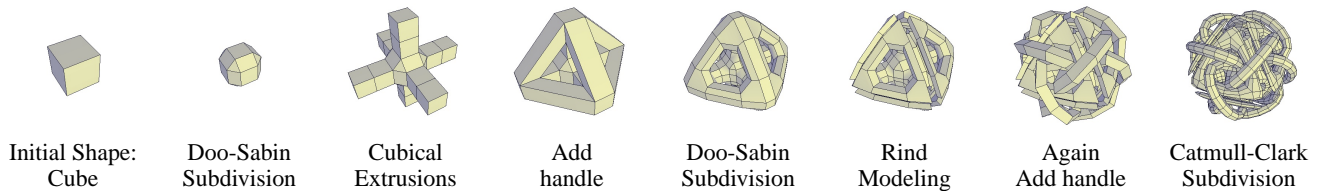


Figure 7: The Procedure used by Cem Yuksel and final virtual sculptures.

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