Animation of Mathematical Concepts using Polynomiography

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Abstract

In this paper we demonstrate how a medium called *polynomiography*, which consists of techniques for visualization of polynomial equations, can be used to animate mathematical concepts, thereby offering a valuable tool for education. More specifically, we will show how it can be used to visualize the following topics: Voronoi regions of points in the plane; multiplication of complex numbers and their interpretation as rotation; sensitivity of polynomial roots as coefficients change; visualization of classes of special polynomial equations arising from two problems from the American Mathematical Monthly; as well as animation for the sake of visual art. Each of these will be exhibited through a series of images and we give the Internet links to the corresponding animations. These sites can be accessed for educational purposes and will be upgraded and expanded from time to time.

CR Categories: G.1.5 [Numerical Analysis]: Roots of Nonlinear Equations—Polynomials, method for; G.4 [Mathematical Software]: Algorithm design and analysis; K.3.1 [Computers and Education]: Computer-assisted instruction; I.3.3 [Computer Graphics]: Picture/Image Generation—Viewing algorithms; J.5 [Arts and Humanities]: Fine arts

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1 Introduction

It is not surprising that a visual image or an animation can sometimes be much more effective in conveying certain concepts or properties than pages of explanations. Together, of course, these would provide the most effective combination. Perhaps in no other field do we see the validity of this claim so easily than the field of mathematics. The purpose of this paper is to demonstrate this property through a medium called *polynomiography*, defined to be "the art and science of visualization in the approximation of zeros of complex polynomials, via fractal and non-fractal images created using the mathematical convergence properties of iteration functions" (see [Kalantari 2002b; Kalantari 2003b; Kalantari 2003a; Kalantari 2004b] for more detail).

A *complex polynomial* of degree n is an expression of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where the coefficients a_i are *complex numbers*. A complex number is a number of the form $z = x + iy$, where x, y are real and $i = \sqrt{-1}$. The *set of complex numbers* is denoted by $\mathbf{C}$. The *modulus* of z, denoted by $|z|$ is defined by $\sqrt{x^2 + y^2}$. The polar representation of z is (r, θ), where $r = |z|$ and θ is the angle that the vector (x, y) forms with the positive x-axis. In particular, $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$.

The problem of approximating roots of $p(z)$, i.e. the solutions to $p(z) = 0$, is a fundamental and classic problem (see [McNamee 1993]). Only polynomials of degree $n \leq 4$ have a closed formula for computing their roots. But even the problem of computing square-roots have to be carried out through approximation means, the best known of which is Newton's method, defined for the general polynomial by the iteration function

$$N(z) = z - \frac{p(z)}{p'(z)}.$$

The challenge of visualization of polynomial root-finding behavior via Newton's method was attempted before the advent of modern computer. In 1879, Cayley [Cayley 1897] raised questions on the behavior of Newton's method for quadratic and cubic polynomials in the complex plane (see [Peitgen and Richter 1996]). He was only able to find the answer for quadratics, where the regions of attraction are merely the Voronoi regions of the two roots. Given a general complex polynomial $p(z)$, the Voronoi region of a particular root θ is a convex polygon defined by the locus of points which are closer to this root than to any other root. The region of attraction for a root θ is the set of points in the plane such that when used as a starting point Newton's iterations will converge to θ. For cubic polynomials, the regions of attraction of Newton's method give only a crude approximation of the actual Voronoi regions of the three roots. The boundaries of these regions exhibit fractal behavior and are now known as *Julia* sets. The computer visualization of this phenomenon was apparently first obtained by John Hubbard (see [Glick 1988]).

More generally, one can consider visualization of polynomial root-finding, i.e. polynomiography, via other iteration functions. In particular, an infinite family of iteration functions called the *Basic Family* [Kalantari and Kalantari 1996; Kalantari et al. 1997; Kalantari 1999; Kalantari 2000a; Kalantari 2000b] can be used. Polynomiography provides two-dimensional images of the process of approximation, viewed through this infinite family of iteration functions. An individual image is called a *polynomiograph*. The word "polynomiography" is a combination of the word *polynomial* and the suffix *-graphy*. A polynomiograph may or may not turn out to be a fractal image.

The Basic Family, whose individual members include the well-known Newton and Halley methods, have been rediscovered independently by other researchers through different means. It is a powerful family which admits numerous different forms and representations. Schröder [Schröder 1870] apparently is one of the first to have derived and studied this family in some generality. The same family is also sometimes referred to as König's family (see [Vrscay and Gilbert 1988]). In fact the Basic Family and more advanced versions are closely related to the celebrated Taylor's Theorem and the Fundamental Theorem of Algebra (see [Kalantari 2000a]).

The Basic Family is denoted by $B_m(z)$, $m = 2, 3, \cdots$. The first member of the Basic Family, $B_2(z)$, is Newton's iteration function, and $B_3(z)$ is Halley's iteration function, which dates back to 1694.

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The rich history of these two iteration functions can be found in [Ypma 1995] and [Traub 1964]. Many results on the properties of the members of the Basic Family, including their close tie with a determinantal generalization of Taylor's theorem, can be found in [Kalantari and Kalantari 1996; Kalantari et al. 1997; Kalantari 1999; Kalantari 2000a; Kalantari and Gerlach 2000; Kalantari and Park 2001; Kalantari 2000b; Kalantari and Jin 2003; Kalantari 2004a].

A convenient form for the Basic Family members is through a remarkably compact closed formula. Consider $p(z)$. Set $D_0(z) \equiv 1$, and for each natural number $m \geq 1$, define

$$D_m(z) = \sum_{i=1}^{n} (-1)^{i-1} \frac{p^{i-1}(z)p^{(i)}(z)}{i!} D_{m-i}(z), \quad D_j = 0, \quad j < 0.$$

For each $m \geq 2$, define

$$B_m(z) \equiv z - p(z) \ \frac{D_{m-2}(z)}{D_{m-1}(z)}.$$

Now if θ is a simple root of $p(z)$ (i.e. $p'(\theta) \neq 0$), then there exists a neighborhood of θ such that given any a_0 in this neighborhood, the fixed-point iteration $a_{k+1} = B_m(a_k)$ is well-defined and converges to θ having order m (roughly speaking the number of correct decimals grow by a factor of m in each iteration).

The basin of attraction of a root θ of $p(z)$ with respect to the iteration function $B_m(z)$ is the set of points in the complex plane such that, when used as the initial point a_0, the corresponding sequence of fixed-point iterates, $a_{k+1} = B_m(a_k)$, will converge to θ. It turns out that the boundary of the basin of attraction of θ coincides with the corresponding set of any other root θ' of $p(z)$. This common boundary is known as a Julia set and its complement as a Fatou set. The fractal nature of Julia sets and the images of the basins of attraction of Newton's method have become familiar after Mandelbrot's work (see The Fractal Geometry of Nature [Mandelbrot 1983]) popularized the work of Julia [Julia 1918] and Fatou [Fatou 1919]. Analysis of fractals was later undertaken by [Peitgen et al. 1992].

We can use the Basic Family either *individually*, or *as a sequence*, to visualize polynomial equations. Given any input a, the Basic Sequence $\{B_m(a) = a - p(a)D_{m-2}(a)/D_{m-1}(a)\}_{m=2}^{\infty}$ converges to a root of $p(z)$. Under some regularity assumptions (e.g. simplicity of the roots), for almost all inputs within the Voronoi polygon of a root, the corresponding Basic Sequence converges to that root.

While the theoretical aspects of polynomiography do intersect with both the theory of fractals and dynamical systems, it is claimed that polynomiography has its own independent characteristics and existence. In fact, polynomiography will not only help produce a unified perspective into the theory of root-finding, but will also enable the discovery of new properties of this ancient problem. Polynomiography is perhaps the most systematic method for the visualization of root-finding algorithms – bringing it to the realm of art and design (see [Kalantari 2002b; Kalantari 2003b; Kalantari 2003a; Kalantari 2004b] for more detail). In this paper we are interested in educational aspects of polynomiography. In particular, applications that benefit from animation through polynomiography. We will exhibit polynomiography's utility through a set of examples and give the Internet links to the corresponding animations.

2 Animation of Approximate Voronoi Regions

The Voronoi region of a root θ of $p(z)$ is a convex polygon defined by the locus of points which are closer to this root than to any other root. More precisely, the Voronoi region of a root θ is

$$V(\theta) = \{z \in \mathbf{C} : |z - \theta| < |z - \theta'|, \ \ \theta' \in R_p, \ \ \theta' \neq \theta\}.$$

Each Voronoi region is a polygon which may or may not be bounded. The boundary of each Voronoi region is either a line segment or a ray consisting of points equidistant to two distinct roots of the polynomial. Any finite set of points in the plane corresponds to the set of roots of a polynomial equation, and conversely. The Basic Family has the property that for large m the basins of attraction provide close approximation of the Voronoi regions (as m approaches infinity they converge to the Voronoi regions). Although the Voronoi regions of a given set of points can be computed very efficiently using computational geometry techniques, if the set of points is given as a polynomial equation then polynomiography provides a direct approach for computing the Voronoi regions without the need to compute the roots in advance (see also [Kalantari 2002a]). In particular, this becomes desirable when the polynomial is sparse (having a few nonzero coefficients).

Polynomiography is a convenient medium to demonstrate or discover Voronoi regions. Figure 1 shows the polynomiography of $z^4 - 1$ and Figure 2 the polynomiography of a polynomial corresponding to a random set of points.

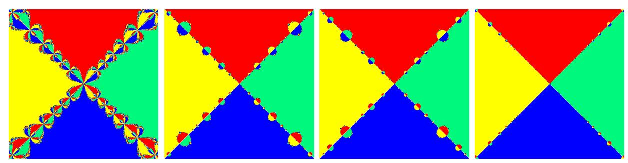

Figure 1: Evolution of basins of attraction to Voronoi regions via $B_m(z)$: $p(z) = z^4 - 1$, $m = 2,3,4,50$. For animation visit `www.cs.rutgers.edu/~kalantar/Animation`, Voronoi Regions of Roots of Unity.

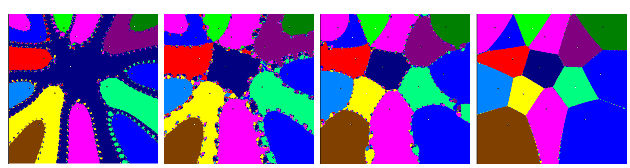

Figure 2: Evolution of basins of attraction to Voronoi regions via $B_m(z)$: random points, $m = 2,4,10,50$. For animation visit `www.cs.rutgers.edu/~kalantar/Animation`, Voronoi Regions of Random Points.

3 Animation of Root Sensitivity

It is well known that the roots of polynomials maybe *sensitive* to small changes in their coefficients. Classical example is the polynomial

$$p(z) = (z-1)(z-2)\cdots(z-n).$$

For instance, for $n=7$ we have:

$$p(z) =$$

$$z^7 - 28z^6 + 322z^5 - 1960z^4 + 6769z^3 - 13132z^2 + 13068z - 5040.$$

Even changing the coefficient of z^6 from -28 to, say, -28.002 causes a somewhat large change in the roots; indeed some real roots become complex. This phenomenon can be visualized via polynomiography. Figure 3 shows a few instances corresponding to gradual changes in the coefficient of z^6.

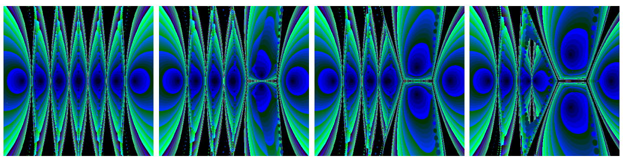

Figure 3: Changes in the roots as the coefficient of z^6 is decreased. For animation visit `www.cs.rutgers.edu/~kalantar/Animation`, Voronoi Region of Random Points.

4 Animation of Complex Multiplication

Indeed polynomiography is a rich medium for teaching the properties of complex numbers. Consider two complex numbers $z_1 = a+ib$ and $z_2 = c+id$. Their addition is defined as $(a+b)+i(c+d)$, while their product is defined as $(ac-bd)+i(ad+bc)$. If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then their product is also $r_1r_2e^{i(\theta_1+\theta_2)}$. This means their product is the complex number whose modulus is the product of the two moduli while its angle can be interpreted as that of rotating the vector (a,b) counterclockwise by the angle θ_2, if θ_2 is positive; or clockwise by the angle $-\theta_2$, if θ_2 is negative. This property can be viewed through polynomiography as follows.

Consider the polynomial $p(z) = c(z-z_1)\cdots(z-z_n)$. Let γ be a complex number. Consider the polynomial $p(\gamma z)$. Clearly, the roots of $p(\gamma z)$ are the solutions to

$$\gamma z - z_i = 0, \quad i = 1,\cdots,n.$$

Thus the new roots are

$$z_i' = z_i/\gamma.$$

Hence, the roots of $p(\gamma z)$ are those of $p(z)$ multiplied by the complex number $1/\gamma$. But if $\gamma = re^{i\theta}$, then $1/\gamma = r^{-1}e^{-i\theta}$. This means the roots of $p(\gamma z)$ will be rotated, clockwise or counterclockwise by the angle θ, while their moduli will be scaled by the factor r^{-1}.

As an example, if we take $p(z) = z^4 - 1$, and $\gamma = e^{i\pi/3}$, then we expect that the roots of $p(z)$ will be rotated by an angle of 60 degrees clockwise while their magnitudes remain unchanged. We can actually visualize this via polynomiography. We have

$$p(\gamma z) = (e^{i\pi/3})^4 z^4 - 1 = e^{4\pi i/3} z^4 - 1.$$

Visualizing $p(z)$ and $p(\gamma z)$ is captured in Figure 4.

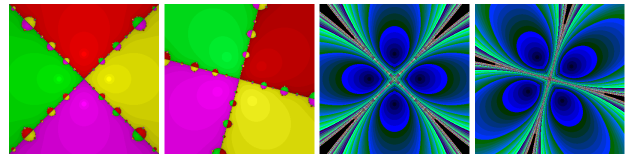

Figure 4: Two polynomiographies of $p(z) = z^4 - 1$ and $p(e^{i\pi/3}z)$. For animation visit `www.cs.rutgers.edu/~kalantar/Animation`, Rotation.

5 Animation of Polynomials Arising in a Problem of Knuth

Let $a = (a_0, a_1, \ldots, a_n)$ be an $(n+1)$-vector where $a_k \in [0,1]$, $k = 0,\ldots,n$. Consider the polynomial

$$p_a(z) = \sum_{k=0}^{n} a_k \binom{n}{k} z^k (1-z)^{n-k}.$$

An inequality concerning these polynomials appears in the American Mathematical Monthly 110, January 2003, Problem 10985.

Here we are interested in how the roots of $p_a(z)$ depend on the vector a. When $a_k = 1$ for all k, then $p_a(z) = (z+1-z)^n \equiv 1$. For each vertex of the $(n+1)$-dimensional hypercube B^{n+1} we obtain a polynomial $p_a(z)$. One possible experimentation in understanding the nature of the roots is to consider polynomiography of $p_{a_\alpha}(z)$, where

$$a_\alpha = \alpha u + (1-\alpha)v, \quad \alpha \in [0,1]$$

with u and v vertices of B^{n+1}, i.e. where a is a convex combination of two vertices of the hypercube. Intuitively, we may want u and v to be diagonally opposite of each other, i.e. $u_i = 1 - v_i$, for $i = 0,\ldots,n$.

As an example when $n=1$, $u = (0,1,0)$, and $v = (1,0,1)$, then $a_\alpha = (1-\alpha, \alpha, 1-\alpha)$, and it is easy to show that

$$p_{a_\alpha}(z) = (2-4\alpha)z^2 - (2-4\alpha)z + 1 - \alpha.$$

For a polynomiography of this visit `www.cs.rutgers.edu/~kalantar/Animation`, Polynomiography of polynomials in a problem of Knuth.

6 Animation of a Problem from the Monthly

The following is posed as problem 10987 in the American Mathematical Monthly 110, January 2003 issue:

Consider the polynomial

$$P(z) = z^t(z-z_1)\cdots(z-z_n),$$

where $|z_i| \geq 1$. Show that $p'(z)$ has no zeros in the disc $\{z : |z| < t/(t+n)\}$.

We argue that this can be verified via polynomiography for some special polynomials. One way to verify this is to consider Newton's method. If Newton's method is well-defined at a point z, which is not a root of $p(z)$, then $p'(z)$ cannot be zero.

Consider the simple case where $p(z) = z^t(z^n - 1)$. Then,

$$p'(z) = z^{t-1}((t+n)z^n - t).$$

So, when $z \neq 0$ is in the disc $\{z : |z| < (t/(t+n))^{1/n}\}$, then $p'(z) \neq 0$.

Figure 5 gives a polynomiography of this for $p(z) = z^t(z^4 - 1)$, $t = 1, 2, 3$. Note the large region of convergence for Newton's method for the root at origin. This is depicted as the red region in the figure.

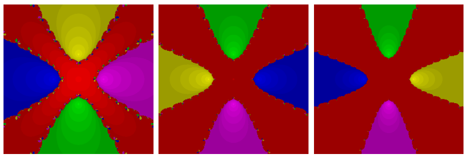

Figure 5: Polynomiography of $p(z) = z^t(z^4 - 1)$. For animation visit `www.cs.rutgers.edu/∼kalantar/Animation`, A Problem of the Monthly.

Figure 6 gives a polynomiography of such a $p(z)$, $t = 1, 2, 6$, but having random nontrivial roots. The green region is Newton's region of convergence to the origin.

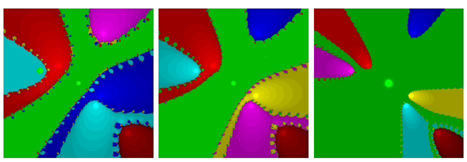

Figure 6: Polynomiography of a random polynomial with nontrivial roots outside of unit disc.

7 Animations as Visual Art

It is possible to use polynomiography as a tool for creating interesting animation as visual art. Here we simply refer the reader to the Internet site `www.cs.rutgers.edu/∼ kalantar/Animation`. This aspect of polynomiography in itself has promising applications and will be exploited more extensively elsewhere.

8 Conclusion

In this paper we have demonstrated how polynomiography can be used to animate mathematical properties, concepts, and even theorems, thereby offering a revealing tool for educators. It appears to be a practical and significant educational instrument with diverse applications. It can be used at various levels: not only at high schools and middle schools, but as a tool that can even entice children interested in mathematics and polynomials. The fact that all animations in this article were implemented by a high school student - Aleksei Andreev - while using a polynomiography software of the first author – speaks well that such applications are quite possible.

On the other hand, polynomiography is a sophisticated tool that can be used at college level by students and teachers. Indeed scientists too could find polynomiography to be a useful tool. Artistic applications of animation with polynomiography are invitingly a field worthy of serious attention. In this paper we have only given a glimpse of what may be possible through polynomiography animation. The true animation possibilities of polynomiography are indeed vast. In achieving some of the advanced educational or artistic applications, it is desirable to develop more sophisticated polynomiography software that is also user-friendly for the particular audience or purpose being considered. This is a future plan as is providing further material concerning polynomiography and its applications.

References

CAYLEY, A. 1897. The Newton-Fourier imaginary problem. *American Journal of Mathematics 2*, 97.

FATOU, P. 1919. Sur les équations fonctionelles. *Bull. Soc. Math. France 47*, 161–271.

GLICK, J. 1988. *Chaos: Making a New Science*. Penguin Books.

JULIA, G. 1918. Sur l'itération des fonctions rationnelles. *J. Math. Pure Appl. 4*, 47–245.

KALANTARI, B., AND GERLACH, J. 2000. Newton's method and generation of a determinantal family of iteration functions. *J. of Comp. and Appl. Math. 116*, 195–200.

KALANTARI, B., AND JIN, Y. 2003. On extraneous fixed-points of the basic family of iteration functions. *BIT 43*, 453–458.

KALANTARI, B., AND KALANTARI, I. 1996. High order iterative methods for approximating square roots. *BIT 36*, 395–399.

KALANTARI, B., AND PARK, S. 2001. A computational comparison of the first nine members of a determinantal family of root-finding methods. *J. of Comp. and Appl. Math. 130*, 197–204.

KALANTARI, B., KALANTARI, I., AND ZAARE-NAHANDI, R. 1997. A basic family of iteration functions for polynomial root finding and its characterizations. *J. of Comp. and Appl. Math. 80*, 209–226.

KALANTARI, B. 1999. On the order of convergence of a determinantal family of root-finding methods. *BIT 39*, 96–109.

KALANTARI, B. 2000a. Generalization of Taylor's theorem and Newton's method via a new family of determinantal interpolation formulas and its applications. *J. of Comp. and Appl. Math. 80*, 287–318.

KALANTARI, B. 2000b. On homogeneous linear recurrence relations and approximation of zeros of complex polynomials. In *Unusual Applications in Number Theory*, AMS, DIMACS Series in Discrete Mathematics and Theoretical, DIMACS, To Appear.

KALANTARI, B. 2002a. Can polynomiography be useful in computational geometry? *DIMACS Workshop on Computational Geometry, Rutgers University (http://dimacs.rutgers.edu/Workshops/CompGeom/program.html)*.

KALANTARI, B. 2002b. Polynomiography: A new intersection between mathematics and art. *Technical Report DCS-TR-506, Department of Computers Science, Rutgers University, New Brunswick, NJ (http://www.polynomiography.com)*.

KALANTARI, B. 2003a. The art in polynomiography of special polynomials. In *Proceedings of ISAMA/BRIDGES.*

KALANTARI, B. 2003b. Polynomiography and applications in art, education, and science. In *Proceedings of SIGGRAPH, Educators Program*.

KALANTARI, B. 2004a. An infinite family of bounds on the zeros of analytic functions and relationship to Smale's bound. *Math. of Comp.*, To Appear.

KALANTARI, B. 2004b. Polynomiography and applications in art, education, and science. *Computers and Graphics*, To Appear.

MANDELBROT, B. B. 1983. *The Fractal Geometry of Nature*. W.F. Freeman.

MCNAMEE, J. M. 1993. A bibliography on roots of polynomials. *Comp. and Appl. Math. 47*, 391–394.

PEITGEN, H.-O., AND RICHTER, H. P. 1996. *The Beauty of Fractals*. Springer.

PEITGEN, H.-O., SAUPE, D., JURGENS, H., AND YUNKER, L. 1992. *Chaos and Fractals*. Springer-Verlag.

SCHRÖDER, E. 1870. On infinitely many algorithms for solving equations (german) (english translation by g.w. stewart, tr-92-121, institute for advanced computer studies, university of maryland, college park, md, 1992.). *Math. Ann. 2*, 317–365.

TRAUB, J. F. 1964. *Iterative Methods for the Solution of Equations*. Prentice Hall.

VRSCAY, E., AND GILBERT, W. 1988. Extraneous fixed points, basin boundaries and chaotic dynamics for Schröder and König iteration functions. *Numer. Math. 52*, 1–16.

YPMA, T. J. 1995. Historical development of Newton-Raphson method. *SIAM Review 37*, 531–551.