Polynomiography and Applications in Art, Education, and Science

Bahman Kalantari
Department of Computers Science
Rutgers University
New Brunswick, NJ, USA

Abstract

Polynomiography is the art and science of visualization in approximation of zeros of complex polynomials. Informally speaking polynomiography allows one to take colorful pictures of polynomials. These images can subsequently be re-colored in many ways using one's own creativity and artistry. It has tremendous applications in visual arts, education, and science. The paper describes some of these applications. From the artistic point of view polynomiography can be used to create quite a diverse set of images reminiscent of the intricate patterning of carpets and elegant fabrics; abstract expressionist and minimalist art; and even images that resemble cartoon characters. From the educational point of view polynomiography can be used to teach mathematical concepts, theorems, and algorithms, e.g. the algebra and geometry of complex numbers; the notions of convergence, and continuity; geometric constructs such as Voronoi regions; and modern notions such as fractals. From the scientific point of view it provides not only a tool for viewing polynomials, present in virtually every branch of science, but also a tool to discover new theorems.

CR Categories: G.1.5 [Numerical Analysis]: Roots of Nonlinear Equations—Polynomials, method for; G.4 [Mathematical Software]: Algorithm design and analysis; K.3.0 [Computers and Education]: General; I.3.3 [Computer Graphics]: Picture/Image Generation—Viewing algorithms; J.5 [Arts and Humanities]: Fine arts

Keywords: Scientific Visualization, Weird Math, Voronoi Regions, Fractals

1 Introduction

Polynomiography is defined to be "the art and science of visualization in approximation of zeros of complex polynomials, via fractal and non-fractal images created using the mathematical convergence properties of an infinite family of iteration functions." An individual image is called a "polynomiograph." The word polynomiography is a combination of the word "polynomial" and the suffix "graphy." These images are obtained using variety of algorithms which require the manipulation of thousands of pixels on a computer monitor. Depending upon the degree of the underlying polynomial it is possible to obtain beautiful images on a laptop computer within minutes. Polynomials form a fundamental class of mathematical objects which arise in virtually every branch of science.

According to the Fundamental Theorem of Algebra, a polynomial of degree n, with real or complex coefficients, has n real or complex zeros (roots) which may or may not be distinct. The problem of approximation of zeros of polynomials, considered even by the Sumerians (third millennium B.C.), has been one of the most influential problems in the development of several important areas in mathematics. Polynomiography is a new approach to solve and view this ancient problem, while making use of new algorithms and today's computer technology. Polynomiography is based on the use of one or an infinite number of iteration functions designed for the purpose of approximation of roots of polynomials. An iteration function is a mapping of the plane into itself, i.e. given any point in the plane, it is a rule that provides another point. An iteration function can be viewed as a machine that approximates a zero of a polynomial by an iterative process that takes an input and from it creates an output which in turn becomes a new input to the same machine.

Polynomiography's foundation is well-defined: visualization in approximation of zeros of polynomials. Using various properties of convergence, as well as variety of coloration schemes, it provides two-dimensional images of the process of approximation, viewed through the lenses of an infinite family of iteration functions. A polynomiograph may or may not turn out to be a fractal image. The word "fractal" coined by the world-renowned research scientist Benoit Mandelbrot refers to sets or geometric objects that are self-similar and independent of scale. Some fractal images can be obtained via simple iterative schemes leading to sets known as Julia set and the famous Mandelbrot set. The simplicity in the creation of such images has resulted in numerous web sites where amateurs and experts exhibit their fractal images. The fact that a polynomiograph may not be fractal image is one of the reasons why it was necessary to define a new name. A second reason is to emphasize its objective and means, i.e. visualization of polynomials via approximation of its roots and the way in which the approximation is carried out. Even when a polynomiograph is a fractal image it does not diminish its uniqueness. Indeed even fractal images obtained from polynomiography reveal more variety, degrees of freedom, and creativity than typical fractal images which are only based on mere iterations of function without a definitive goal in mind.

2 A Brief Foundation

Consider the polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $n \geq 2$, and the coefficients a_i are complex numbers. The problem of approximating roots of $p(z)$ is a fundamental and classic problem. A large bibliography can be found in [McNamee 1993]. For a good article on some history, applications, and algorithms, see [Pan 1997]. See also the book [Blum et al. 1997].

One of the conceptually easiest algorithms for the approximation of all the roots of polynomials is described in [Kalantari 2002b], making use of a fundamental family of iteration functions, studied in [Kalantari et al. 1997]. The family is called the "Basic Family," and is represented as $\{B_m(z)\}_{m=2}^{\infty}$. The first member of the sequence, $B_2(z)$, is Newton's iteration function, and $B_3(z)$ is Halley's iteration function, which dates back to 1694. The rich history of these two iteration functions can be found in [Ypma 1995] and [Traub 1964]. Many results on the properties of the members of the Basic Family, including their close tie with a determinantal generalization of Taylor's theorem can be found in [Kalantari and Kalantari

1996; Kalantari et al. 1997; Kalantari 1999; Kalantari 2000; Kalantari and Gerlach 2000; Kalantari and Park 2000; Kalantari 2002b].

The Basic Family admits several different representations and it has been rediscovered by several authors using various techniques. But in fact the Basic Family and its multipoint versions are all derivable from a determinantal generalization of Taylor's theorem [Kalantari 2000]. A recent result on the Basic Family and additional references may be found at [Kalantari and Jin 2003].

The members of the Basic Family have a beautiful closed formula. Consider $p(z)$. Set $D_0(z) \equiv 1$, and for each natural number $m \geq 1$, define

$$D_m(z) = \det \begin{pmatrix} p'(z) & \frac{p''(z)}{2!} & \dots & \frac{p^{(m-1)}(z)}{(m-1)!} & \frac{p^{(m)}(z)}{(m)!} \\ p(z) & p'(z) & \ddots & \ddots & \frac{p^{(m-1)}(z)}{(m-1)!} \\ 0 & p(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{p''(z)}{2!} \\ 0 & 0 & \dots & p(z) & p'(z) \end{pmatrix}.$$

For each $m \geq 2$, define

$$B_m(z) \equiv z - p(z) \ \frac{D_{m-2}(z)}{D_{m-1}(z)}.$$

The following theorem describes some of the fundamental properties of the members of the Basic Family. For a complex number $c = a + ib$, where $i = \sqrt{-1}$, its modulus is denoted by $|c| = \sqrt{a^2 + b^2}$.

Theorem 2.1 ([Kalantari 1998; Kalantari 2002b]) *The following conditions hold:*

1. *For all $m \geq 1$ we have,*

$$D_m(z) = \sum_{i=1}^{n} (-1)^{i-1} \frac{p^{i-1}(z) p^{(i)}(z)}{i!} D_{m-i}(z), \quad D_j = 0, \ \ j < 0.$$

2. *Let θ be a simple root of $p(z)$. Then there exists a neighborhood of θ such that given any a_0 in this neighborhood the fixed-point iteration $a_{k+1} = B_m(a_k)$ is well-defined, it converges to θ having order m (i.e. roughly speaking the number of correct decimals grow by a factor of m in each iteration).*

To describe a fundamental global property of the sequence let

$$R_p = \{\theta_1, \dots, \theta_t\} \tag{1}$$

be the set of distinct roots of $p(z)$. The elements of R_p partition the Euclidean plane into Voronoi regions and their boundaries. The Voronoi region of a root θ is a convex polygon defined by the locus of points which are closer to this root than to any other root. More precisely, the Voronoi region of a root θ is

$$V(\theta) = \{z \in \mathbf{C} : |z - \theta| < |z - \theta'|, \ \ \theta' \in R_p, \ \ \theta' \neq \theta\}. \tag{2}$$

Let S_p be the locus of points that are equidistant from two distinct roots, i.e.

$$S_p = \{z \in \mathbf{C} : |z - \theta| = |z - \theta'|, \ \text{ where } \ \theta, \theta' \in R_p, \ \ \theta \neq \theta'\}. \tag{3}$$

This is a set of measure zero consisting of the union of finite number of lines.

Definition 2.2 *Given $a \in \mathbf{C}$ the Basic Sequence at a is defined as*

$$B_m(a) = a - p(a) \frac{D_{m-2}(a)}{D_{m-1}(a)}, \quad m = 2, 3, \dots.$$

Theorem 2.3 ([Kalantari 2002b]) *Given $p(z)$, for any input $a \notin S_p$, the Basic Sequence is well-defined satisfying*

$$\lim_{m \to \infty} B_m(a) = \theta,$$

for some $\theta \in R_p$. Under some regularity assumptions, e.g. simplicity of all the roots of $p(z)$, for all $a \in V(\theta)$, $\lim_{m \to \infty} B_m(a) = \theta$.

2.1 Basins of Attractions of Polynomial Roots

Consider a polynomial $p(z)$ and a fixed natural number $m \geq 2$. The basins of attraction of a root of $p(z)$ with respect to the iteration function $B_m(z)$ are regions in the complex plane such that given an initial point a_0 within them the corresponding sequence $a_{k+1} = B_m(a_k)$, $k = 0, 1, \dots$, will converge to that root. It turns out that the boundary of the basins of attractions of any of the polynomial roots is the same set. This boundary is known as the Julia set and its complement it known as the Fatou set. The fractal nature of Julia sets and the images of the basins of attractions of Newton's method are now quite familiar for some special polynomials, e.g. $p(z) = z^3 - 1$. A fractal image of this polynomial via Newton's method, was apparently first obtained by the American Mathematician John H. Hubbard (see [Glick 1988]). Mandelbrot's work, see The Fractal Geometry of Nature [Mandelbrot 1983], in particular popularized the Julia theory [Julia 1918] on the iteration of rational complex functions and the work of Fatou [Fatou 1919], and led to the famous set that bears Mandelbrot's name. [Peitgen et al. 1992] undertake a further analysis of fractals. Mathematical analysis of complex iterations may be found in [Peitgen and Richter 1996; Devaney 1986], and [Falconer 1990].

While the fractal nature of the Julia sets corresponding to the individual members of the Basic Family follows from the Julia theory on rational iteration function, that theory does not predict the total behavior of specific iteration functions on the complex plane. In contrast there are important consequences of the results stated in the theorems of the previous section which apply to arbitrary polynomials. In particular, Theorem 2.3 implies the following: Except possibly for the locus of points equidistant to two distinct roots, given any input a, the Basic Sequence $\{B_m(a) = a - p(a)D_{m-2}(a)/D_{m-1}(a)\}$ converges to a root of $p(z)$. Under some regularity assumption (e.g. simplicity of the roots), for almost all inputs within the Voronoi polygon of a root, the corresponding Basic Sequence converges to that root. The Basic Sequence corresponds to the pointwise evaluation of the Basic Family.

3 Polynomiography and Visual Art

From the artistic point of view polynomiography is a new art form. Working with a polynomiography software is comparable to working with a camera or a musical instrument. Through practice one can learn to produce the most exquisite, complex, and diverse set of patterns. These designs, at their best, are analogous to the most sophisticated human designs. Polynomiography could be used in classrooms for the teaching of art or mathematics, from children to college level students. It could also be used by professional artists in various ways and as a graphics tool. Even an amateur art lover could learn to use a polynomiography software without any knowledge of mathematics.

The "polynomiographer" can create an infinite variety of designs. This is made possible by employing an infinite variety of iteration functions (which are analogous to the lenses of a camera), to the infinite class of complex polynomials (which are analogous to photographic models). The polynomiographer then may go through the same kind of decision making as the photographer: changing scale, isolating parts of the image, enlarging or reducing, adjusting values and color until the polynomiograph is resolved into a visually satisfying entity. Like a photographer, a polynomiographer can learn to create images that are esthetically beautiful and individual, with or without the knowledge of mathematics or art. Like an artist and a painter, a polynomiographer can be creative in coloration and composition of images. Like a camera, or a painting brush, a polynomiography software can be made simple enough that even a child could learn to operate.

Polynomiography gives rise to a problem which may be called, "reverse root-finding" : Given a polynomial whose roots form a known set of points find an iteration function and coloration scheme which turn the image into a desired or beautiful design.

It is possible to develop a polynomiography software where the user can create images by inputing a polynomial through several means, e.g. by inputing its coefficients, or the location of its zeros. In one approach the user simply inputs a parameter, m, as any natural number greater than one. The assignment of a value for m corresponds to the selection of $B_m(z)$ as the underlying iteration function. This together with the selection of a user-defined rectangular region and user-specified number of pixels, as well as a variety of color mapping schemes gives the capabilities of creating an infinite number of basic polynomiographs. Such polynomiographs turn out to be fractal images and can subsequently be easily re-colored or zoomed in any number of times. A non-fractal and completely different set of images could result by the visualization of the root-finding process for a given polynomial through the collective use of the Basic Family and the pointwise convergence property depicted in Theorem 2.3. These images are enormously rich. In either type of image creation technique the user has a great deal of choices, e.g. the ability to re-color any selected regions using a variety of coloration based on convergence properties.

Viewing polynomiography as an art form, one can list at least four general image creation techniques.

(1) Like a photographer who shoots different pictures of a model and uses a variety of lenses, a polynomiographer can produce different images of the same polynomial and make use of a variety of iteration functions and zooming approaches until a desirable image is discovered.

(2) In this more creative approach an initial polynomiograph, possibly very ordinary, is turned into a beautiful image, based on the user's coloration, individual creativity, and imagination.

(3) The user employs the mathematical properties of the iteration functions, or the underlying polynomial, or both (this is truly a marriage of art and mathematics).

(4) Images can be obtained as a collage of two or more polynomiographs obtained through one of the previous three methods. Many other image creation techniques are possible, either through artistic compositional means, or through computer assisted design programs.

Figure 1-5 presents a sample set of images obtained via a prototype polynomiography software. More detail on the production of these images as well as a larger collection of them may be found at [Kalantari 2002c].

4 Polynomiography and Education

Polynomiography has enormous potential applications in education. A polynomiography software could be used in the mathematics classroom as a device for understanding polynomials as well as the visualization of theorems pertaining to polynomials. As an example of application of polynomiography, high school students studying algebra and geometry could be introduced to mathematics through creating designs from polynomials. They would learn to use algorithms on a sophisticated level and to understand mathematics of polynomials in its relationship to pattern and design in ways that cannot be approached abstractly. Students can visually

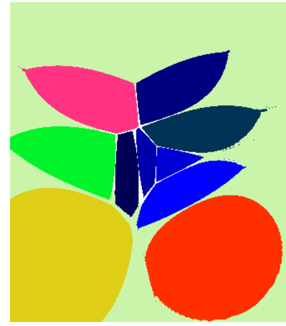

Figure 1: **"Summer"**

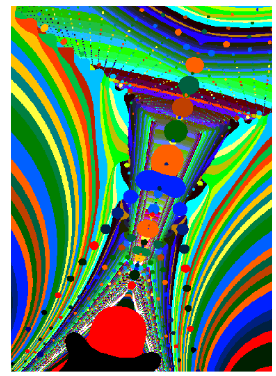

Figure 2: **"Circus"**

discover the Fundamental Theorem of Algebra while playing with a prototype software. It is possible to compile many theorems about polynomials and their properties, or those of iteration functions, visualizable through polynomiography. Indeed using polynomiography students can learn about the algebra and geometry of complex numbers. For instance symmetric designs can be obtained by considering polynomials whose roots have symmetric patterns. One can obtain interesting images by simply considering polynomials whose roots are the roots of unity or more generally n-th root of a real number r. By multiplication of these polynomials, as well as rotation of the roots (which amounts to multiplication by complex numbers) one can obtain some very interesting basic designs that can subsequently be turned into beautiful designs. In the process algebraic operations on complex numbers will be transformed into visualizable geometric objects.

At a higher educational level, e.g. calculus or numerical analysis courses, polynomiography allows students to tackle important conceptual issues such as the notion of convergence and limits, as well as the idea of iteration functions, and gives the student the ability to understand and appreciate more modern discoveries such as fractals.

Another interesting educational application of polynomiography is in teaching concepts within geometry, e.g. Voronoi regions. For instance Figure 6 and Figure 7 present several fractal images that confirm the theoretical convergence results: as m increases, the basins of attractions to the roots, as computed with respect to the iteration function $B_m(z)$, rapidly converge to the Voronoi regions of the roots. Thus the regions with chaotic behavior rapidly shrink to the boundaries of the Voronoi regions. In Figure 6 we consider a polynomial with a random set of roots, depicted as dots. The figure shows the evolution of the basins of attraction of the roots to the Voronoi regions as m takes the values $2, 4, 10$, and 50. Figure

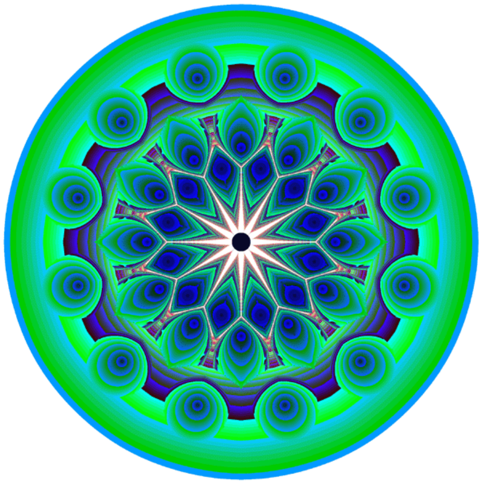

Figure 3: **Polynomiograph of a degree 36 polynomial**

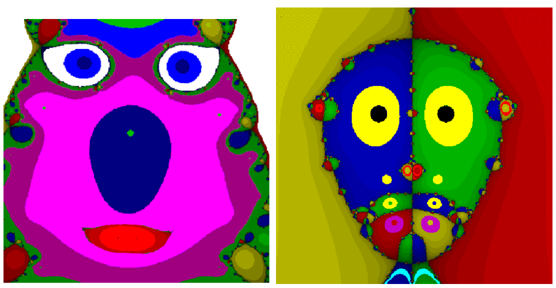

Figure 4: “Ms. Poly” and “L3”

7 shows the basin of attractions for the polynomials $P(z) = z^4 - 1$, corresponding to different values of m. The roots of $p(z) = z^4 - 1$ are the roots of unity and hence the Voronoi regions are completely symmetric. In these figures in the case of $m = 2$, i.e. Newton’s method, the basins of attractions are chaotic. However, these regions rapidly improve by increasing m.

In summary while working with polynomiography software students can learn about a number of beautiful and useful concepts.

5 Polynomiography and Science

Polynomials are undoubtedly the most fundamental class of functions and arise in virtually every branch of science. Essentially a polynomial is completely characterized if one knows all its roots. Thus knowing a polynomial is knowing all the roots. And this together with the universality of polynomials makes polynomiography a very desirable tool for scientists of diverse background. In order to get insight into polynomials one can look at their polynomiography. This visualization, even if carried out for not too large a degree polynomials, could be of tremendous value. When we begin to think about polynomials, even special polynomials e.g. Legendre polynomials, Chebeyshev polynomials, orthogonal polynomials, etc. we may notice that we really don’t have a good picture of them. Polynomiography can engrave a certain visual attributes of each of the special class of polynomials or an individual polynomial that we may come across. For instance Figure 8 gives some polynomiography for a polynomial arising in physics [Cambell and

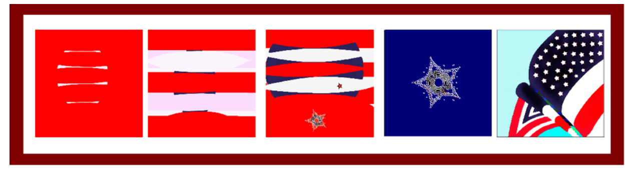

Figure 5: **“Evolution of Stars and Stripes”**

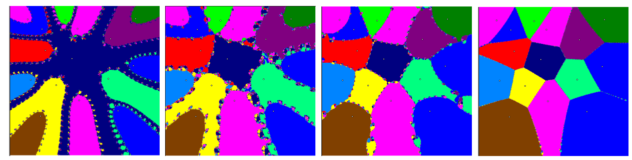

Figure 6: **Evolution of basins of attraction to Voronoi regions via $B_m(z)$: random points,** $m = 2, 4, 10, 50$

Kadtke 1987].

Polynomiography has numerous algorithmic and theoretical applications. For instance in the development of other root-finding algorithms. Polynomiography is not only a means for obtaining good algorithms for polynomial root-finding but allows the users to determine how two particular members of the Basic Family compare. Which is a better method Newton’s method or Halley’s method? Most numerical analysis books do not bother with such questions. But indeed these are fundamental questions from pedagogical and practical point of view. One may make use of polynomiographs to study the computational advantages of members of the Basic Family over Newton’s method, as well as the advantages in using the Basic Sequence in computing a single root, or all the roots of a given polynomial. Through polynomiography it is also possible to discover new properties of polynomials and or iteration functions. These can subsequently be formally proved mathematically. For some potential applications of polynomiography to computational geometry see [Kalantari 2002a].

As a final scientific and/or commercial application of polynomiography we consider the encryption of numbers, e.g. ID numbers or credit card numbers into a two dimensional image that resembles a fingerprint. Different numbers will exhibit different fingerprints. One way to visualize numbers as polynomiographs is to represent them as polynomials. For instance a hypothetical social security number $a_8 a_7 \cdots a_0$ can be identified with the polynomial $P(z) = a_8 z^8 + \cdots + a_1 z + a_0$. Now we can apply any of the techniques discussed earlier. A particularly interesting visualization results when the software makes use of Basic Family collectively. Figures 9 gives an example.

6 Conclusion

In this paper polynomiography has been described and its application with respect to three different areas have been explored: art, education, and science. Clearly, applications of polynomiography within each of these areas demands more analysis and research. We are however convinced that each of these deserve serious attention.

References

BLUM, L., CUCKER, F., SHUB, M., AND SMALE, S. 1997. *Com-*

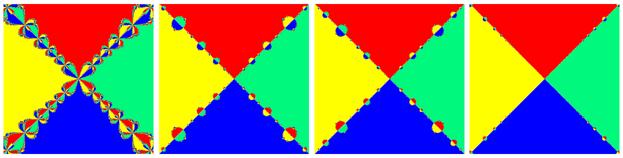

Figure 7: **Evolution of basins of attraction to Voronoi regions via $B_m(z)$: $p(z) = z^4 - 1$, $m = 2, 3, 4, 50$**

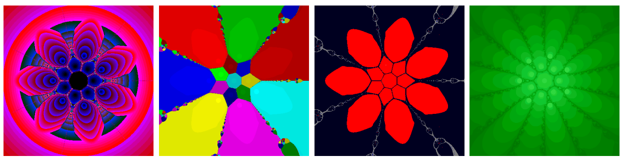

Figure 8: **Some polynomiography of** $45z^{15} + 45z^8 - z$

plexity And Real Computation. Springer.

Cambell, L., and Kadtke, J. 1987. Stationary configuration of point vortices and other logarithmic objects in two dimensions. *PHYSICAL REVIEW LETTERS 58*, 670–673.

Devaney, R. L. 1986. *Introduction to Chaotic Dynamic Systems*. Benjamin Cummings.

Falconer, K. J. 1990. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons.

Fatou, P. 1919. Sur les équations fonctionelles. *Bull. Soc. Math. France 47*, 161–271.

Glick, J. 1988. *Chaos: Making a New Science*. Penguin Books.

Julia, G. 1918. Sur l'itération des fonctions rationnelles. *J. Math. Pure Appl. 4*, 47–245.

Kalantari, B., and Gerlach, J. 2000. Newton's method and generation of a determinantal family of iteration functions. *J. of Comp. and Appl. Math. 116*, 195–200.

Kalantari, B., and Jin, Y. 2003. On extraneous fixed-points of the basic family of iteration functions. *To Appear in BIT 43*.

Kalantari, B., and Kalantari, I. 1996. High order iterative methods for approximating square roots. *BIT 36*, 395–399.

Kalantari, B., and Park, S. 2000. A computational comparison of the first nine members of a determinantal family of root-finding methods. *J. of Comp. and Appl. Math. 130*, 197–204.

Kalantari, B., Kalantari, I., and Zaare-Nahandi, R. 1997. A basic family of iteration functions for polynomial root finding and its characterizations. *J. of Comp. and Appl. Math. 80*, 209–226.

Kalantari, B. 1998. Approximation of polynomial root using a single input and the corresponding derivative values. *Technical Report DCS-TR 369, Department of Computer Science, Rutgers University, New Brunswick, New Jersey*.

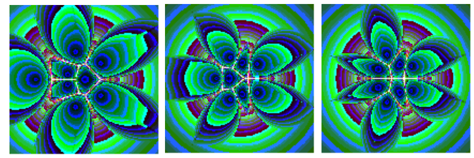

Figure 9: **A polynomiograph of the three numbers 387624730, 945326221,856123201**

Kalantari, B. 1999. On the order of convergence of a determinantal family of root-finding methods. *BIT 39*, 96–109.

Kalantari, B. 2000. Generalization of taylor's theorem and newton's method via a new family of determinantal interpolation formulas and its applications. *J. of Comp. and Appl. Math. 80*, 287–318.

Kalantari, B. 2002. Can polynomiography be useful in computational geometry? *DIMACS Workshop on Computational Geometry, Rutgers University (http://dimacs.rutgers.edu/Workshops/CompGeom/program.html)*.

Kalantari, B. 2002. On homogeneous linear recurrence relations and approximation of zeros of complex polynomials. In *Unusual Applications in Number Theory*, AMS, DIMACS Series in Discrete Mathematics and Theoretical, DIMACS, To Appear.

Kalantari, B. 2002. Polynomiography: A new intersection between mathematics and art. *Technical Report DCS-TR-506, Department of Computers Science, Rutgers University, New Brunswick, NJ (http://www.polynomiography.com)*.

Mandelbrot, B. B. 1983. *The Fractal Geometry of Nature*. W.F. Freeman.

McNamee, J. M. 1993. A bibliography on root of polynomials. *Comp. and Appl. Math. 47*, 391–394.

Pan, V. Y. 1997. Solving a polynomial equation: Some history and recent progress. *SIAM Review 39*, 187–220.

Peitgen, H.-O., and Richter, H. P. 1996. *The Beauty of Fractals*. Springer.

Peitgen, H.-O., Saupe, D., Jurgens, H., and Yunker, L. 1992. *Chaos and Fractals*. Springer-Verlag.

Traub, J. F. 1964. *Iterative Methods for the Solution of Equations*. Prentice Hall.

Ypma, T. J. 1995. Historical development of newton-raphson method. *SIAM Review 37*, 531–551.