

Course Timeline

1. Fundamentals of Fourier Analysis

Duration: 8:30 – 9:15 (45 minutes)

Speaker: Jonas Gomes

2. From Time-Frequency Localization to Wavelets

Duration 9:15 – 10:00 (45 minutes)

Speaker: Jonas Gomes

Coffee Break

Duration: 10:00 – 10:15 (15 minutes)

3. Filter Banks and Wavelets

Duration 10:15 – 11:00 (45 minutes)

Speaker: Luiz Velho

4. Wavelet Design

Duration 11:00 – 12:00 (60 minutes)

Speaker: Luiz Velho

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